R x S^{1}上のCP^{N-1} 模型の格子シミュレーション

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CP^{N-1} sigma model

2D CP^{N-1} model is not only a toy model of QCD, but also effectively describes gauge theory !

- Effective theory on vortex in U(N) + Higgs model is CP^{N-1} Eto, et.al.(05)
- Effective theory on long strings in YM is CP^{N-1} Aharony, Komargodski(13)
- It is also notable that CP^1 describes spin chain systems Haldane(83)

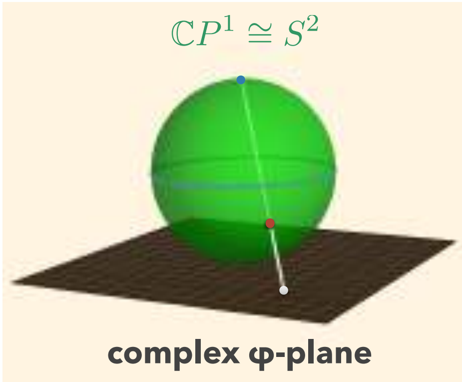
Lattice study on CP^{N-1} model is of physical significance

◆Lagrangian of CP^{N-1} models $\mathbb{C}P^{N-1} = \dfrac{SU(N)}{SU(N-1)\times U(1)}$

$$S = \frac{1}{2g^2}\int |D\phi|^2 \qquad |\phi|^2 = 1,\ D\phi = (d+ia)\phi,\ a = i\bar{\phi}\cdot d\phi$$

discretized on the lattice

$$S = -N\beta\sum_{n,\mu}\left(\bar{z}_{n+\mu}\cdot z_n\lambda_{n,\mu} + \bar{z}_n\cdot z_{n+\mu}\bar{\lambda}_{n,\mu} - 2\right)$$



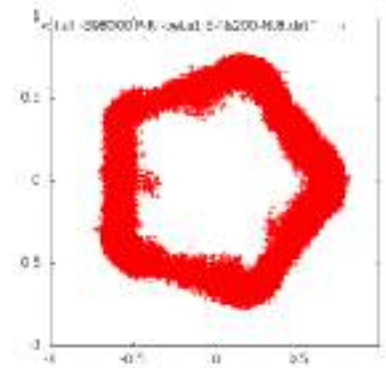
CP^{N-1} sigma model on R x S^1

- Global symmetry : PSU(N) flavor symmetry + Time reversal
- Z_N symmetry is not exact for periodic b. c. (cf. QCD)

- Z_N-twisted b.c.

$$\phi(x_1, x_2 + L) \;=\; \Omega\phi(x_1, x_2) \qquad \Omega = \mathrm{diag.}\left[1, e^{2\pi i/N}, e^{4\pi i/N}, \cdots, e^{2(N-1)\pi i/N}\right]$$

Exact Z_N-symmetry (intertwined of Z_N flavor shift & center)
with Z_N vacua



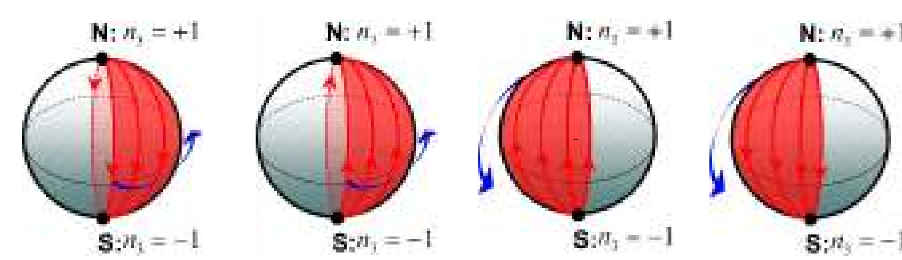
Distribution of P-loop for N=5

- Fractional instantons (Q=1/N, S=S_I/N)

BPS eq. $\quad D\phi \pm \mathrm{i} \star D\phi = 0$

BPS sol. $\quad \phi = \dfrac{(1, e^{2\pi z/(NL)}, \ldots)}{\sqrt{1 + |e^{2\pi z/(NL)}|^2 + \ldots}}$

cf.) N=2



Lee, Yi(97) Kraan, van Baal(97)
Eto, et.al. (04~) Bruckmann,et.al. (05~)

＊It is shown to have resurgent structure (pert. vs non-pert. relation)

Dunne, Unsal(12) TM, Nitta, Sakai(14,15) Fujimori, et.al.(16~)

摂動級数と非摂動的寄与の関係

$$\sum_{q=0} a_q g^{2q} \quad \not\longleftrightarrow \quad \exp\left[-\frac{A}{g^2}\right]$$

摂動級数　　　　　　**非摂動的寄与**

「摂動的寄与と非摂動的寄与は関連付かない異なる寄与」
というのが一般的な見方

本当にそうだろうか？

摂動計算とボレル和

$$\left[H_0 + g^2 H_{\rm pert}\right]\psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$

摂動級数（漸近級数と仮定）は
一般に階乗発散し収束半径0　$a_q \propto q!$

高次まで摂動計算を行っても意味のある情報は得られなさそうだが...

➡ **ボレル変換**：**有限の収束半径を持つ級数に変換**
ボレル和：**元の摂動級数を漸近級数として持つ解析関数**

摂動計算とボレル和

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摂動級数（漸近級数と仮定）は
一般に階乗発散し収束半径0 $a_q \propto q!$

$$\Rightarrow \quad BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

ボレル変換

$$\Rightarrow \quad \mathbb{B}(g^2) = \int_0^\infty \frac{dt}{g^2} e^{-t/g^2} BP(t)$$

ボレル和

摂動計算とボレル和

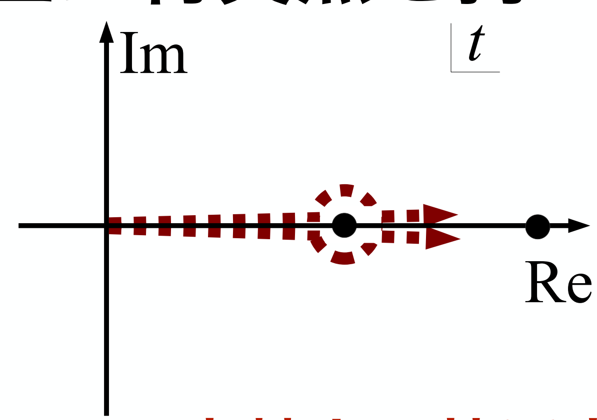
$$\left[H_0 + g^2 H_{\mathrm{pert}}\right]\psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$

摂動級数（漸近級数と仮定）は
一般に階乗発散し収束半径0　$a_q \propto q!$

一般的にはボレル変換が正の実軸上に特異点を持つ

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$



$$\Rightarrow \quad \mathbb{B}(g^2 e^{\pm i\epsilon}) \int_0^{\infty e^{\pm i\epsilon}} \frac{dt}{g^2} e^{-\frac{t}{g^2}} BP(t)$$

正実軸上の特異点のため
積分路に不定性

摂動計算とボレル和

$$\left[H_0 + g^2 H_{\rm pert}\right]\psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$

摂動級数（漸近級数と仮定）は
一般に階乗発散し収束半径0　$a_q \propto q!$

$$\mathbb{B}(g^2 e^{\pm i\epsilon}) = \mathrm{Re}[\mathbb{B}] \pm i\mathrm{Im}[\mathbb{B}]$$

$$\mathrm{Im}[\mathbb{B}(g^2)] \approx e^{-\frac{A}{g^2}}$$

積分路の不定性に付随して
符合の不定性を持つ虚部が出現

この摂動ボレル和の不定虚部こそ非摂動寄与の情報を含む！

常微分方程式のリサージェンス構造

Ecalle (81)

$$\varphi'(z) + \varphi(z) = \frac{1}{z}$$

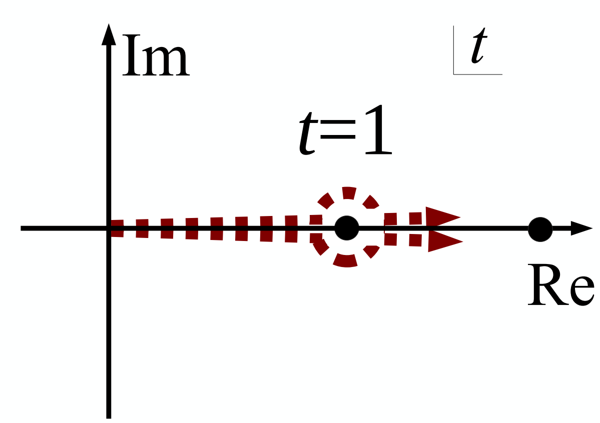
$$z \sim \frac{1}{g^2}$$

形式的解 at $z=\infty$ ⇨ $\Phi_0 = \sum_{q=0}^{\infty} n! z^{-n-1}$ & e^{-z}

摂動的　　非摂動的

$$\mathcal{S}_+\Phi_0(z) - \mathcal{S}_-\Phi_0(z) = 2\pi i e^{-z}$$

ボレル和を通して関係づいている！



リサージェンス構造が存在．しかしなぜ？

常微分方程式のリサージェンス構造

◆オイラー方程式

$$\varphi'(z) + \varphi(z) = \frac{1}{z} \quad \Rightarrow \quad \text{形式的解} \quad \Phi_0 = \sum_{q=0}^{\infty} n! z^{-n-1} \quad \& \quad e^{-z}$$

- 常微分方程式の解は各漸近級数の**ボレル和の総和 = トランス級数**

$$\varphi(z;\sigma) = \Phi_0 + \sigma e^{-z} \xrightarrow{\text{ボレル和}} \mathcal{S}_{\pm}\varphi(z;\sigma) = \mathcal{S}_{\pm}\Phi_0(z) + \sigma e^{-z}$$

- arg[z]=0で**トランス級数パラメタ σ が不連続 = ストークス現象**

$$\mathcal{S}_{+}\varphi(z;\sigma) \to \mathcal{S}_{-}\varphi(z;\sigma+\mathfrak{s})$$

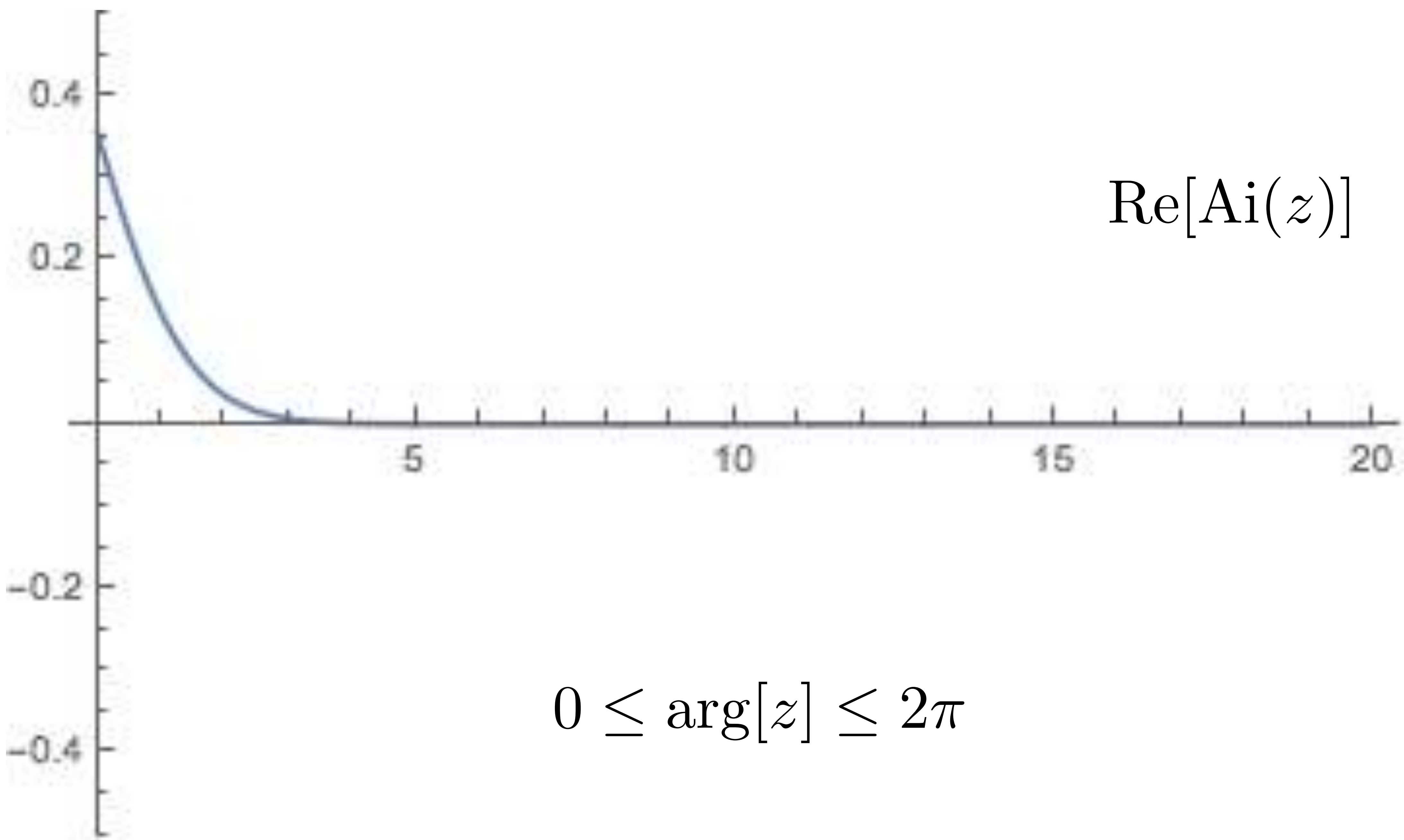
$\mathfrak{s} = 2\pi i$: ストークス定数

- 解の連続性から**各形式的解が結びつく**

$$\mathcal{S}_{+}\varphi(z;\sigma) = \mathcal{S}_{-}\varphi(z;\sigma+\mathfrak{s}) \quad \Rightarrow \quad \mathcal{S}_{+}\Phi_0(z) - \mathcal{S}_{-}\Phi_0(z) = 2\pi i e^{-z}$$

ex.) エアリー方程式 $\varphi'' - z\varphi = 0$ (z = ∞に不確定特異点)

$$\varphi = \mathrm{Ai}(z) \approx \underline{e^{-\frac{2}{3}z^{\frac{3}{2}}}\mathcal{S}_{\pm}\sum a_n z^{-\frac{3}{2}n}} + \sigma\, \underline{e^{\frac{2}{3}z^{\frac{3}{2}}}\mathcal{S}_{\pm}\sum b_n z^{-\frac{3}{2}n}}$$



$\mathrm{Re}[\mathrm{Ai}(z)]$

$0 \leq \arg[z] \leq 2\pi$

積分におけるリサージェンス構造

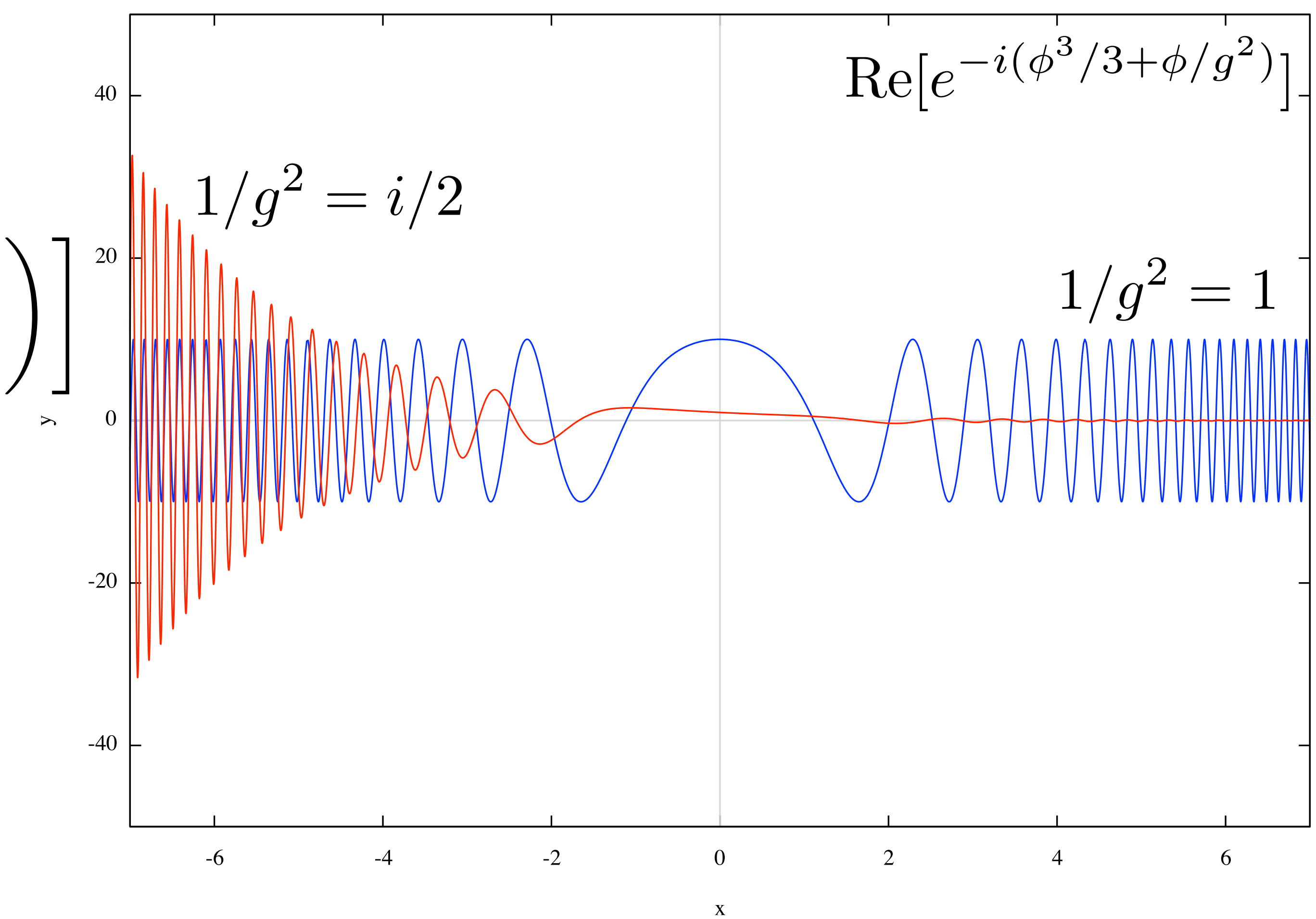
0次元積分における最急降下法では積分経路を変形し
複素固定点に繋がる経路(thimble)に分解

経路積分においても複素固定点を考えるのは自然

・エアリー積分

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

$$\approx \sqrt{\frac{g}{4\pi}} \exp\left(-\frac{2}{3g^2}\right)$$



積分におけるリサージェンス構造

最急降下法(Thimble分解)における複素固定点の寄与

・エアリー積分

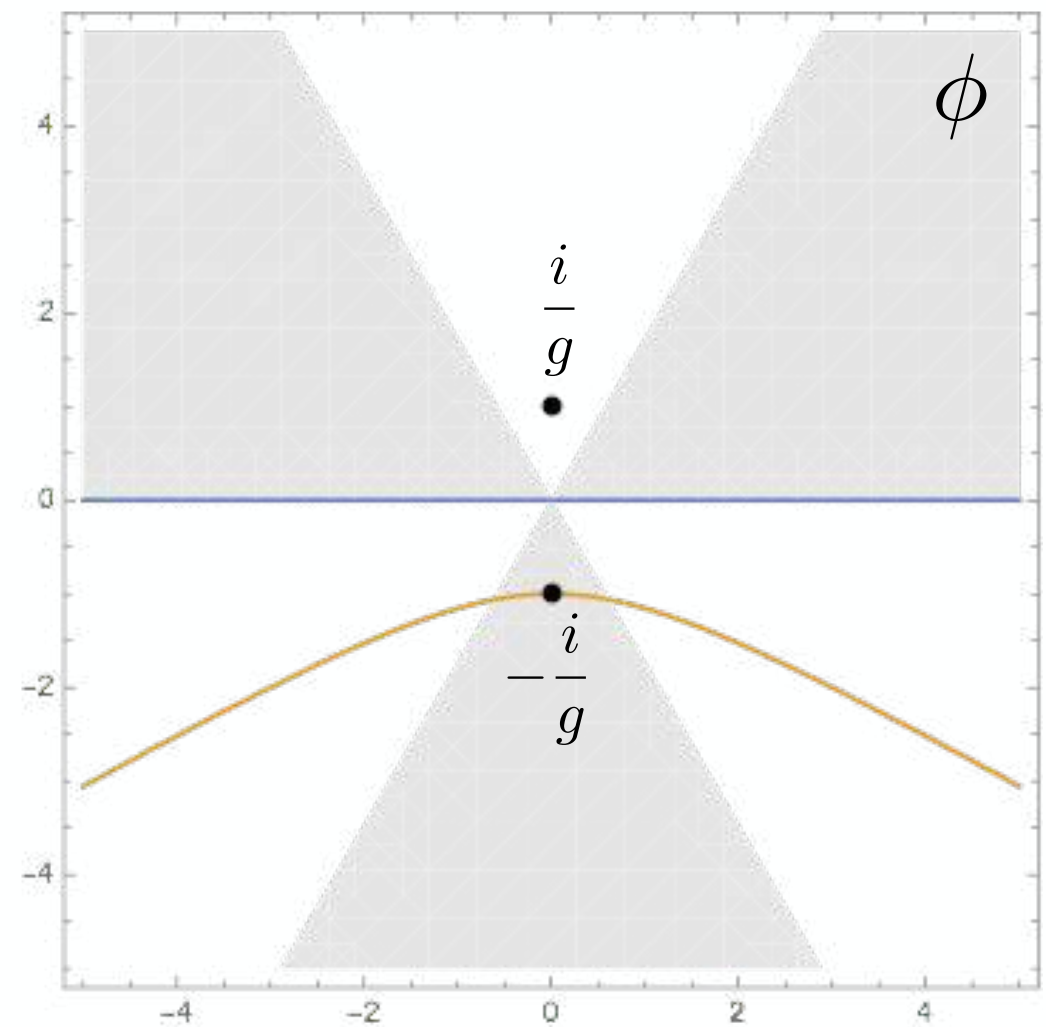
$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3}+\frac{\phi}{g^2}\right)\right]$$

複素平面上の2つの複素固定点 $\phi = \pm\frac{i}{g}$

最急降下法：元の積分径路を，固定点を通り，虚部一定の最急降下径路に分解

$$\mathcal{C} = \sum_{\sigma} n_\sigma \mathcal{J}_\sigma$$

最急降下径路分解
= Thimble分解



$\arg[g^2] = 0+$

積分におけるリサージェンス構造

最急降下法(Thimble分解)における複素固定点の寄与

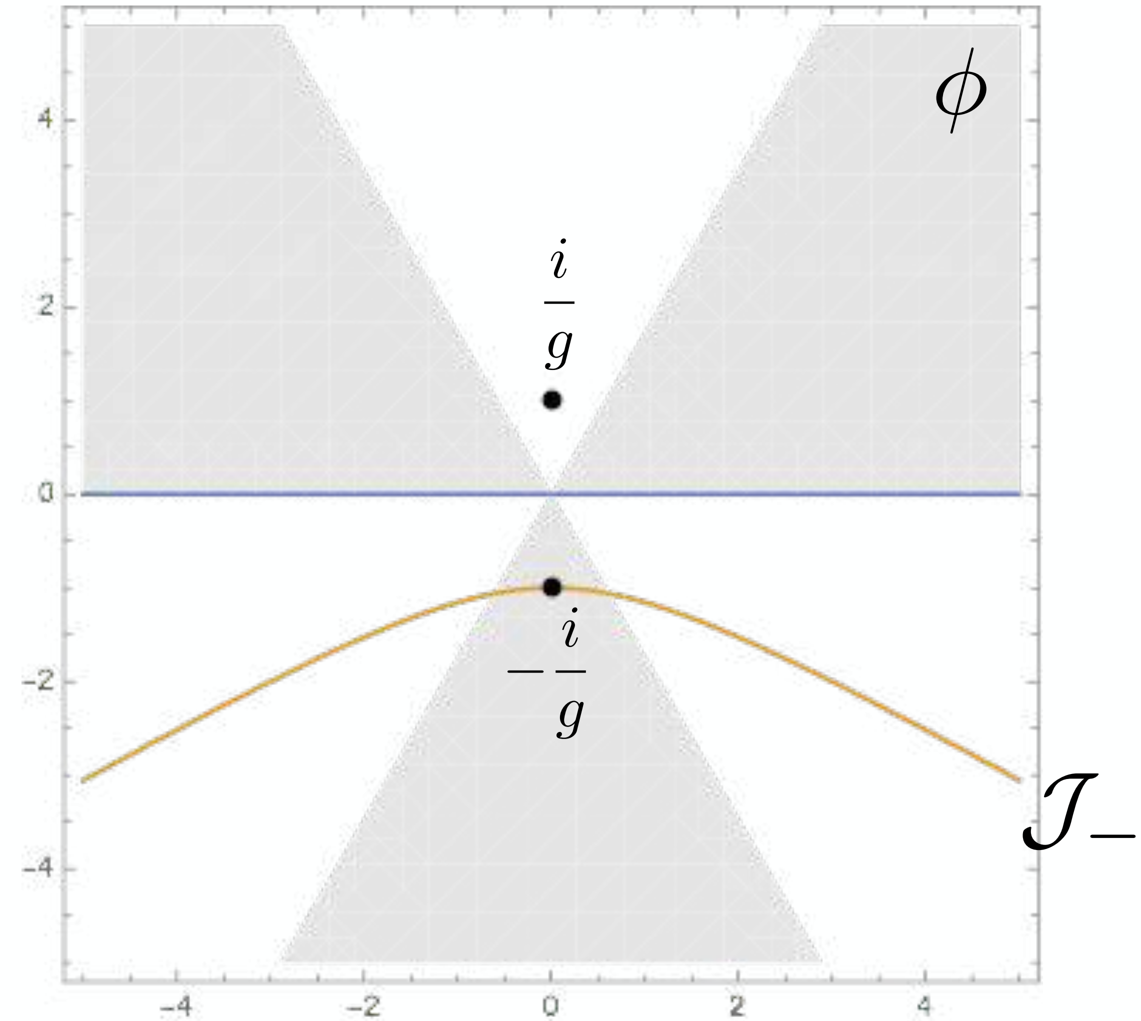
・エアリー積分

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3}+\frac{\phi}{g^2}\right)\right]$$

- $\mathcal{J}_\sigma$ $\quad \mathrm{Im}[S] = \mathrm{Im}[S_0]$, $\mathrm{Re}[S] \leq \mathrm{Re}[S_0]$ 最急降下経路

- $n_\sigma = \langle \mathcal{K}_\sigma, \mathcal{C} \rangle$ 最急上昇経路$\mathcal{K}$と元の経路との交叉数

$$\Rightarrow \quad \mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma$$



$\arg[g^2] = 0+$

積分におけるリサージェンス構造

最急降下法(Thimble分解)における複素固定点の寄与

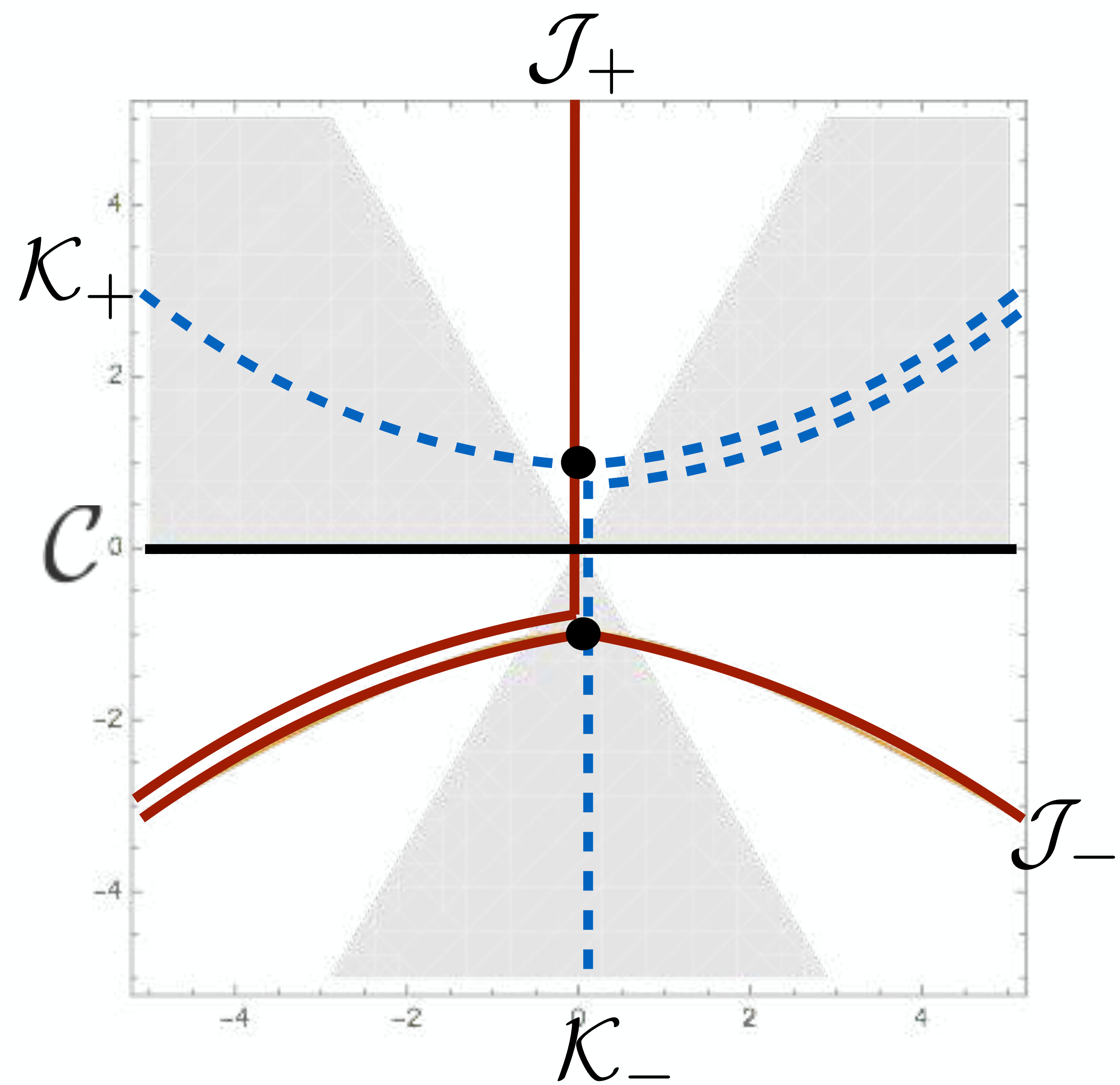
・エアリー積分　$\arg[g^2]=0+$

$$n_+=\langle \mathcal{K}_+,\mathcal{C}\rangle=0$$

$$n_-=\langle \mathcal{K}_-,\mathcal{C}\rangle=1$$

$$\mathcal{C}=\sum_\sigma n_\sigma \mathcal{J}_\sigma \Rightarrow \boxed{\mathcal{C}=\mathcal{J}_-}$$

$\arg[g^2]=\frac{2\pi}{3}-$　まで有効な分解



$\arg[g^2]=0+$

積分におけるリサージェンス構造

最急降下法(Thimble分解)における複素固定点の寄与

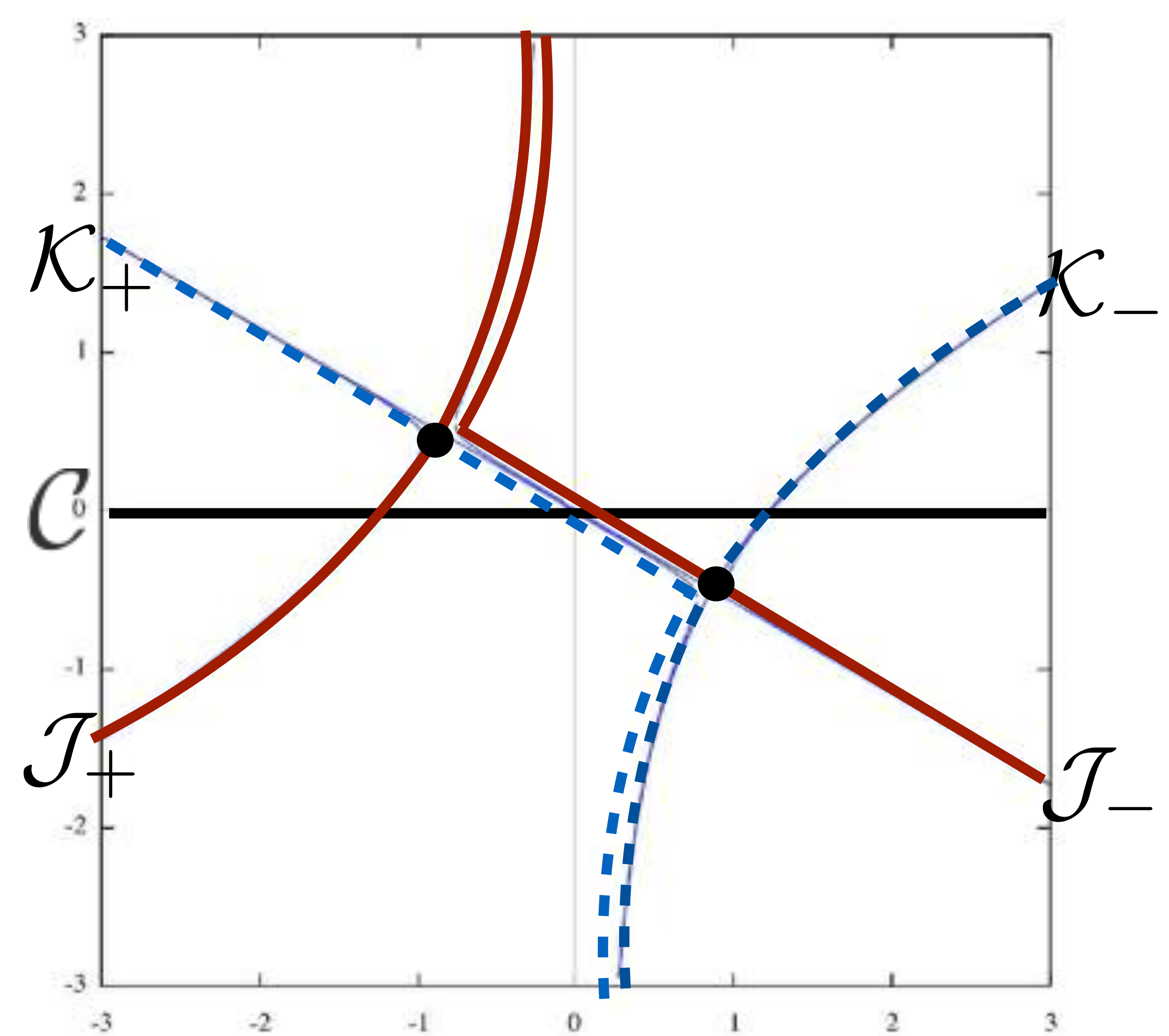
・エアリー積分　$\arg[g^2] = \frac{2\pi}{3} +$

$$n_+ = \langle \mathcal{K}_+, \mathcal{C} \rangle = 1$$

$$n_- = \langle \mathcal{K}_-, \mathcal{C} \rangle = 1$$

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \Rightarrow \boxed{\mathcal{C} = \mathcal{J}_+ + \mathcal{J}_-}$$

ストークス現象：特定のarg[g^2]で thimble分解が不連続変化



$\arg[g^2] = \frac{2\pi}{3} +$

積分におけるリサージェンス構造

最急降下法(Thimble分解)における複素固定点の寄与

・エアリー積分

$$\arg[g^2] = \frac{2\pi}{3}- \qquad\qquad \arg[g^2] = \frac{2\pi}{3}+$$

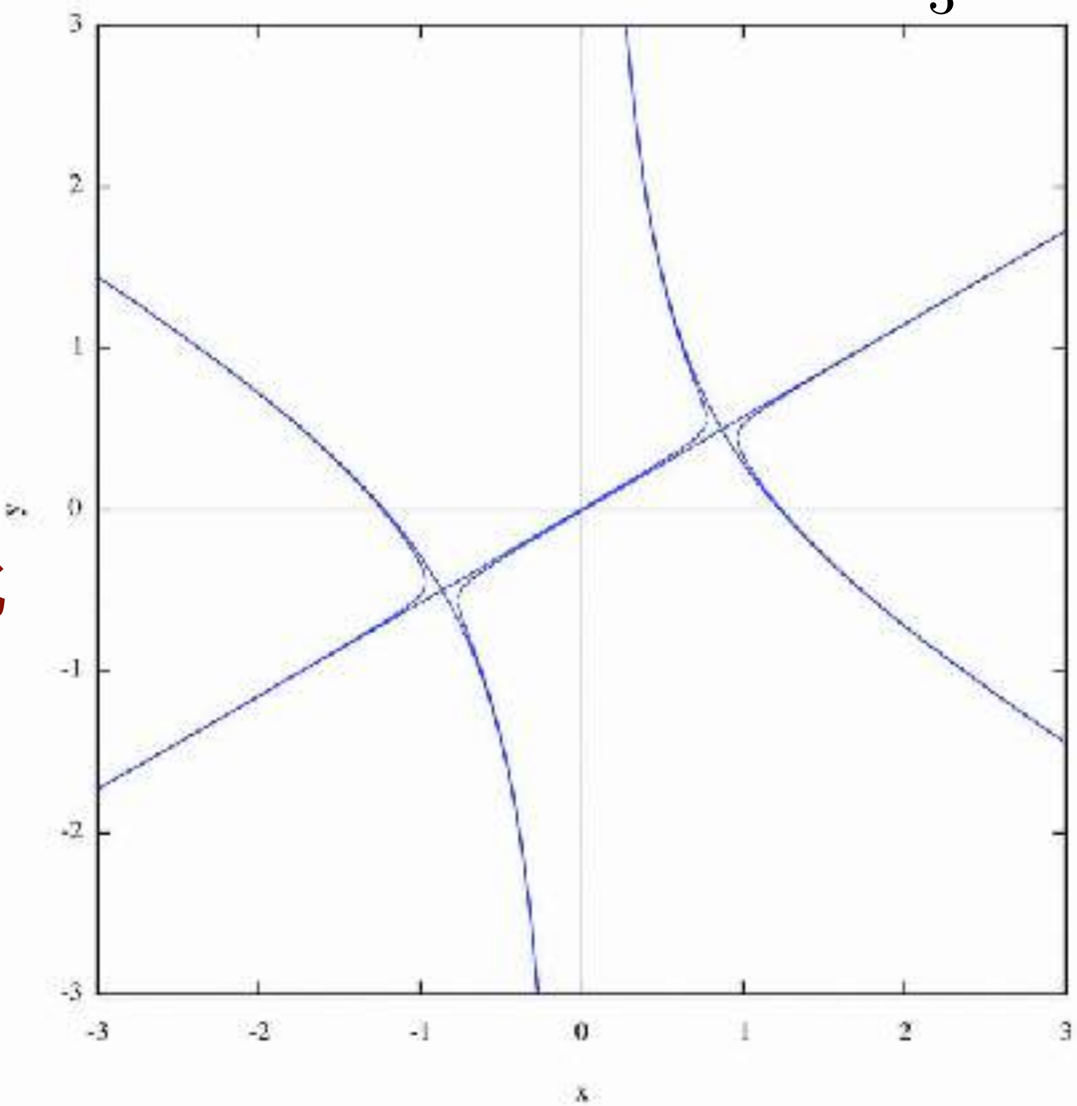
$$\boxed{\mathcal{C} = \mathcal{J}_-} \quad \Leftrightarrow \quad \boxed{\mathcal{C} = \mathcal{J}_+ + \mathcal{J}_-}$$

＊ **Thimble分解がストークス線で不連続に変化**

＊ **エアリー関数自体はストークス線でも連続**

$$\mathcal{J}_-\left[\frac{2\pi}{3}^-\right] = \mathcal{J}_-\left[\frac{2\pi}{3}^+\right] + \mathcal{J}_+$$

ストークス現象 $\arg[g^2] = -\frac{2\pi}{3} \to \pi$



積分におけるリサージェンス構造

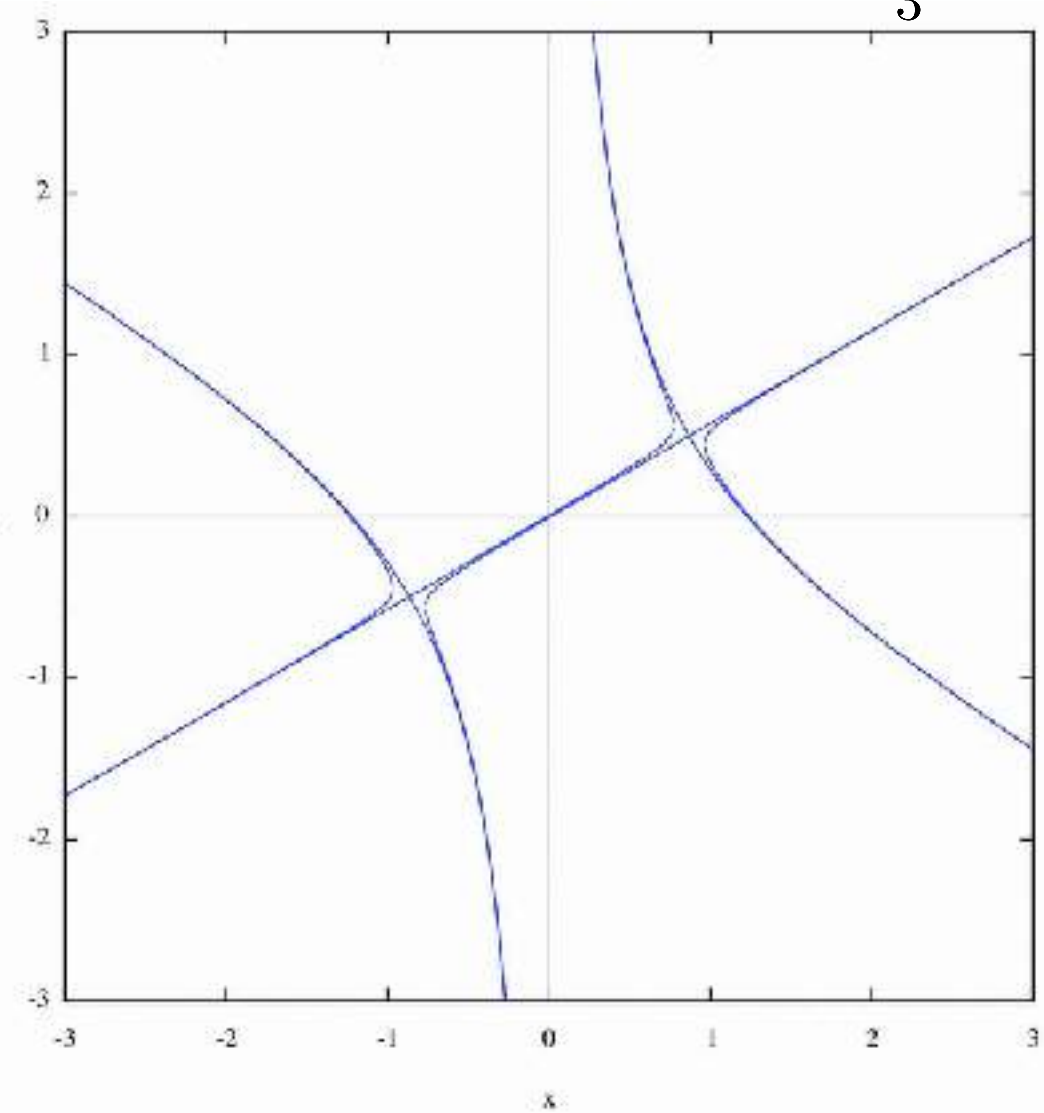
最急降下法(Thimble分解)における複素固定点の寄与

・エアリー積分

$$\arg[g^2] = \frac{2\pi}{3}- \qquad\qquad \arg[g^2] = \frac{2\pi}{3}+$$

$$\boxed{\mathcal{C} = \mathcal{J}_-} \Longleftrightarrow \boxed{\mathcal{C} = \mathcal{J}_+ + \mathcal{J}_-}$$

ストークス現象 $\arg[g^2] = -\frac{2\pi}{3} \to \pi$



＊ **Thimble分解がストークス線で不連続に変化**

＊ **エアリー関数自体はストークス線でも連続**

摂動ボレル和の不定性はストークス線上

でのthimble分解の不定性に対応！

完全なリサージェンス構造 in twisted CP1

・CP1 ハミルトニアン

[Fujimori, Kamata, TM, Nitta, Sakai(16)]

$$H = -g^2(1+\varphi\bar{\varphi})^2 \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\bar{\varphi}} + V(\varphi\bar{\varphi})$$

- $\epsilon = 1$ **SUSY case**

$$\Psi_0 = \exp\left(\frac{m}{2g^2}\frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}}\right) \Rightarrow H\Psi = 0$$

厳密波動関数

ゼロ基底状態エネルギー

Witten Index ≠ 0

完全なリサージェンス構造 in twisted CP1

・CP1 ハミルトニアン [Fujimori, Kamata, TM, Nitta, Sakai(16)]

$$H = -g^2(1+\varphi\bar{\varphi})^2 \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\bar{\varphi}} + V(\varphi\bar{\varphi})$$

- $\epsilon \approx 1$ **near-SUSY case**

$$\delta H = H - H_{\epsilon=1} = -\delta\epsilon m \frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}} \quad \Rightarrow \quad E = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle} + \frac{\langle \delta\psi|\delta H|\delta\psi\rangle}{\langle 0|0\rangle} + \mathcal{O}(\delta\epsilon^3)$$

ハミルトニアンのずれ

$$\delta\epsilon = \epsilon - 1$$

ノンゼロ基底状態エネルギー

完全なリサージェンス構造 in twisted CP1

[Fujimori, Kamata, TM, Nitta, Sakai(16)(17)]

- 基底状態エネルギー　$\delta\epsilon = \epsilon - n$

$$E \;=\; E^{(1)}\delta\epsilon \;+\; E^{(2)}\delta\epsilon^2 \;+\; \mathcal{O}(\delta\epsilon^3)$$

$$E^{(1)} = g^2 - m\coth\frac{m}{g^2} = -m + g^2 - \sum_{p=0}^{\infty} 2me^{-\frac{2pm}{g^2}}$$

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right) + \mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right]$$

摂動寄与と非摂動寄与が含まれるはず

完全なリサージェンス構造 in twisted CP1

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right]$$

- 摂動寄与

$$E_0^{(2)} \approx g^2 - 2m\sum_{n=1}^{\infty}(n-1)!\left(\frac{g^2}{2m}\right)^n$$

ボレル和を実行 arg[*g*^2] = ±0

- 固定点寄与

$$E_{\mathrm{np}}^{(2)} \approx -2m\sum_{p=1}^{\infty}e^{-\frac{2mp}{g^2}}\left[(p+1)^2\sum_{n=1}^{\infty}(n-1)!\left(\frac{g^2}{2m}\right)^n + (p-1)^2\sum_{n=1}^{\infty}(n-1)!\left(-\frac{g^2}{2m}\right)^n - 2p^2\left(\gamma+\log\frac{2m}{g^2}\right)\right]$$

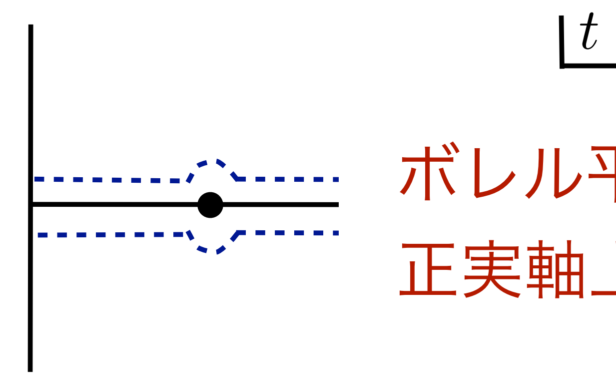
ボレル和を実行 arg[*g*^2] = ±0

完全なリサージェンス構造 in twisted CP1

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right]$$

- 摂動寄与

$$E_0^{(2)} \;=\; g^2 + 2m\int_0^\infty dt\frac{e^{-t}}{t-\frac{2m}{g^2\pm i0}}$$



ボレル平面上の
正実軸上の特異点

- 固定点寄与

$$E_p^{(2)} = 2m\int_0^\infty dt e^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}}+\frac{(p-1)^2}{t+\frac{2m}{g^2}}\right] + 4mp^2\left(\gamma+\log\frac{2m}{g^2}\pm\frac{i\pi}{2}\right)$$

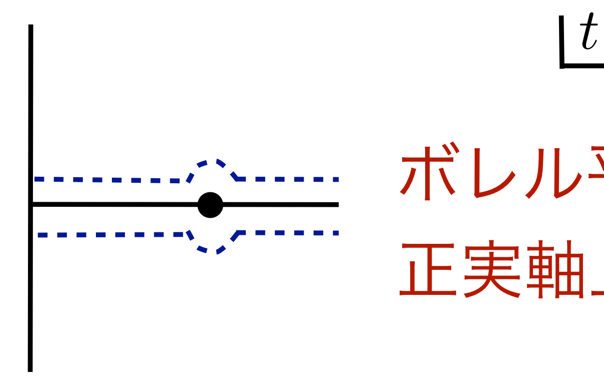
完全なリサージェンス構造 in twisted CP1

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right]$$

- **摂動寄与**

$$E_0^{(2)} \;=\; g^2 + 2m\int_0^\infty dt\frac{e^{-t}}{t-\frac{2m}{g^2\pm i0}}$$

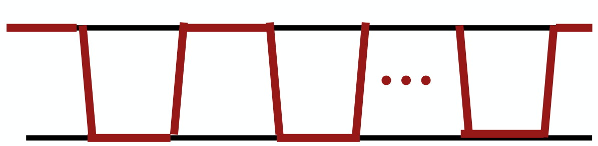
0-bion背景での摂動的寄与



ボレル平面上の
正実軸上の特異点

- **固定点寄与**

$$E_p^{(2)} = 2m\int_0^\infty dte^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}}+\frac{(p-1)^2}{t+\frac{2m}{g^2}}\right] + 4mp^2\left(\gamma+\log\frac{2m}{g^2}\pm\frac{i\pi}{2}\right)$$



***p*-bion背景での摂動的寄与**

***p*-bionの半古典的寄与**

p個の正反instanton対 = p-bion

完全なリサージェンス構造 in twisted CP1

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right]$$

- 摂動寄与

$$E_0^{(2)} \;=\; g^2 + \boxed{2m\int_0^\infty dt\frac{e^{-t}}{t-\frac{2m}{g^2\pm i0}}} \longrightarrow \mp 2mi\pi$$

摂動的寄与の不定虚部は，1-bionの半古典的寄与の不定虚部と相殺

p=1 bion

- 固定点寄与

$$E_p^{(2)} = 2m\int_0^\infty dte^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}} + \frac{(p-1)^2}{t+\frac{2m}{g^2}}\right] + 4mp^2\left(\gamma + \log\frac{2m}{g^2}\boxed{\pm\frac{i\pi}{2}}\right)$$

完全なリサージェンス構造 in twisted CP1

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right]$$

- 摂動寄与

$$E_0^{(2)} = g^2 + 2m\int_0^\infty dt\frac{e^{-t}}{t-\frac{2m}{g^2\pm i0}}$$

- 固定点寄与

(*p*-1)-bion　　相殺　　*p*-bion

$$E_p^{(2)} = \boxed{2m\int_0^\infty dt e^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}}\right.}+\left.\frac{(p-1)^2}{t+\frac{2m}{g^2}}\right] + 4mp^2\left(\gamma + \log\frac{2m}{g^2}\boxed{\pm\frac{i\pi}{2}}\right)$$

(*p*-1)-bion背景での摂動的寄与の不定虚部は，*p*-bionの半古典的寄与の不定虚部と相殺！

完全なリサージェンス構造 in twisted CP1

$$E^{(2)} = g^2 - m\frac{\coth\frac{m}{g^2}}{\sinh^2\frac{m}{g^2}}\left[\frac{\mathrm{Ei}\left(\frac{2m}{g^2}\right)+\mathrm{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log\frac{2m}{g^2}\right]$$

- 摂動寄与

$$E_0^{(2)} \;=\; g^2 + 2m\int_0^\infty dt\frac{e^{-t}}{t-\frac{2m}{g^2\pm i0}}$$

Bender-Wu法による高次摂動係数

$$A_l \sim -\frac{1}{2^{l-1}}\frac{\Gamma(l+2(1-\epsilon))}{\Gamma(1-\epsilon)^2}$$

摂動寄与と一致！

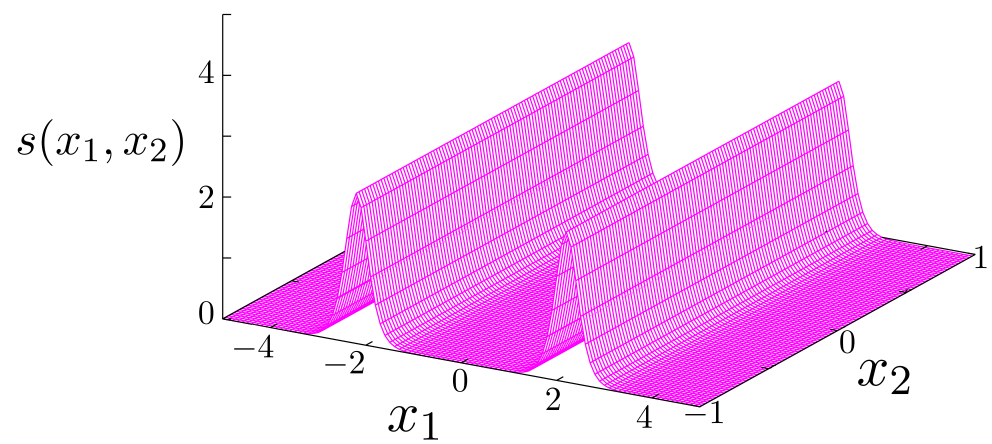
- 固定点寄与

$$E_p^{(2)} = 2m\int_0^\infty dte^{-t}\left[\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}}+\frac{(p-1)^2}{t+\frac{2m}{g^2}}\right]+4mp^2\left(\gamma+\log\frac{2m}{g^2}\pm\frac{i\pi}{2}\right)$$

複素*p*-bion解の寄与と一致！

繰り込みまで含めたリサージェンス構造

[Fujimori, Kamata, TM, Nitta, Sakai(18)]



コンパクト化方向の
依存性はない！

揺らぎを取り入れる
前の有効作用

$$S(x_r, \phi_r) = \frac{4\pi m L}{g^2} - \frac{8\pi m L}{g^2} \cos\phi_r e^{-m x_r} + 2 m x_r$$

ゼータ関数を用いて
KK modeの足し上げを行う

繰り込まれた
有効作用

$$S_R(x_r, \phi_r) = \frac{4\pi m L}{g_R^2} - \frac{8\pi m L}{g_R^2} \cos\phi_r e^{-m x_r} + 2 m x_r$$

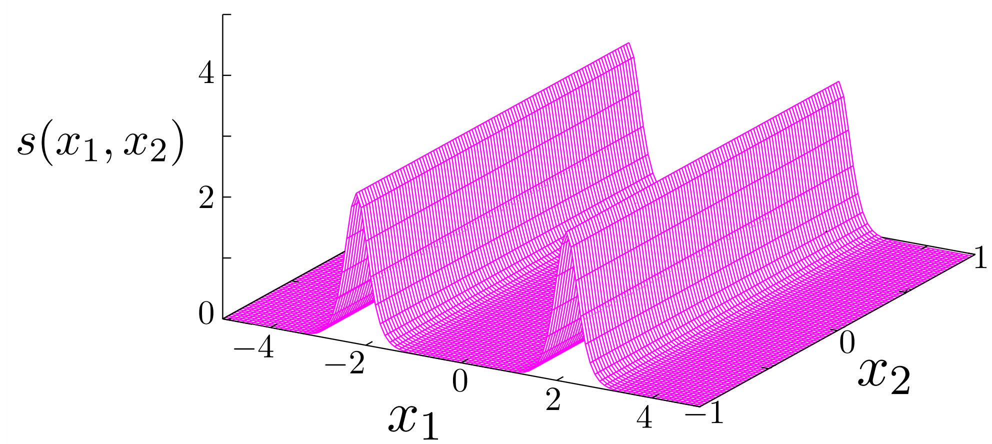
$$\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{1}{\pi} \log L\Lambda_0$$

$$\Lambda = \Lambda_0 e^{-\frac{\pi}{g^2}}$$

力学的スケール

繰り込みまで含めたリサージェンス構造

[Fujimori, Kamata, TM, Nitta, Sakai(18)]



コンパクト化方向の
依存性はない！

揺らぎを取り入れる
前の有効作用

$$S(x_r, \phi_r) = \frac{4\pi mL}{g^2} - \frac{8\pi mL}{g^2}\cos\phi_r e^{-mx_r} + 2mx_r$$

ゼータ関数を用いて
KK modeの足し上げを行う

Bion寄与

$$Z_{bion} \approx |L\Lambda|^{4mL}$$

赤外リノーマロンに
完全に一致！

場の理論の繰り込みも含めて不定虚部の相殺が示された例

Resurgent structure in QM and QFT

$$\mathcal{S}_+\Phi_0(z) - \mathcal{S}_-\Phi_0(z) \approx \mathfrak{s}e^{-Az}\mathcal{S}\Phi_1(z) \qquad z = \frac{1}{g^2}$$

Perturbative imaginary ambiguity

Non-perturbative effect

⇨ **In a certain class of QFT as twisted CP^{N-1} models QFT can be defined based on the structure.**

問題は，コンパクト化半径の大小の間で相転移が生じずに，
上記の構造が保たれるか否か.

Anomaly matching for CP^{N-1} models

Komargodski, Sharon, Thorngren, Zhou (17)

Let us look into mixed 't Hooft anomaly between

flavor $SU(N)/\mathbb{Z}_N$ time reversal T

- Plan : gauge the flavor symmetry and do T transformation
- we first gauge SU(N), then find Z_N 1-form symmetry should be gauged.

$$\mathcal{Z}_\theta[(A,B)] = \int \mathcal{D}a\mathcal{D}\vec{z} \exp\left[-\int \mathrm{d}^2x \left(\frac{1}{2}|(\partial_\mu + \mathrm{i}a_\mu - \mathrm{i}A_\mu)\vec{z}|^2 + V(|\vec{z}|^2)\right) + \frac{\mathrm{i}\theta}{2\pi}\int(\mathrm{d}a + B)\right]$$

$$\mathcal{Z}_\pi[\mathsf{T}\cdot(A,B)]\exp\left(-\mathrm{i}k\int \mathsf{T}\cdot B\right) = \mathcal{Z}_\pi[(A,B)]\exp\left(-\mathrm{i}k\int B\right)\mathrm{e}^{\mathrm{i}(2k-1)\int B}$$

For even N, it has a mixed 't Hooft anomaly

For odd N, we find global inconsistency between $\theta = 0, \pi$

⇨ It suggests spontaneous breaking of T

Z_N-twisted CP^{N-1} models at $\theta = \pi$ on $R \times S^1$

Tanizaki,TM, Sakai (17)

Question: does this anomaly survive for compactified theory ?

- Introducing $\mathbf{Z}_N$ twisted boundary condition on S^1

➡ we have an intertwined Z_N 0-form **shift symmetry** $(\mathbb{Z}_N)_S$

$$\underline{\vec{z} \mapsto S\vec{z}} \quad \& \quad \underline{\Omega \to \omega\Omega}$$

flavor rotation Z_N 0-form transf.

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad S\Omega S^{-1} = \omega\Omega$$

Z_N 0-form shift symmetry $(Z_N)_S$ ➡ U(1) 1-form gauge field $B^{(1)}$

$$\begin{aligned} \mathcal{Z}_{\pi,\Omega}[\mathsf{T}\cdot B^{(1)}] &= \mathcal{Z}_{\pi,\Omega}[B^{(1)}]\exp\left(-\mathrm{i}\int B^{(1)}\int_0^L L^{-1}\mathrm{d}x^2\right) \\ &= \mathcal{Z}_{\pi,\Omega}[B^{(1)}]\exp\left(-\mathrm{i}\int B^{(1)}\right). \end{aligned}$$

Mixed 't Hooft anomaly of $(Z_N)_S$ and T survives in compactified theory !

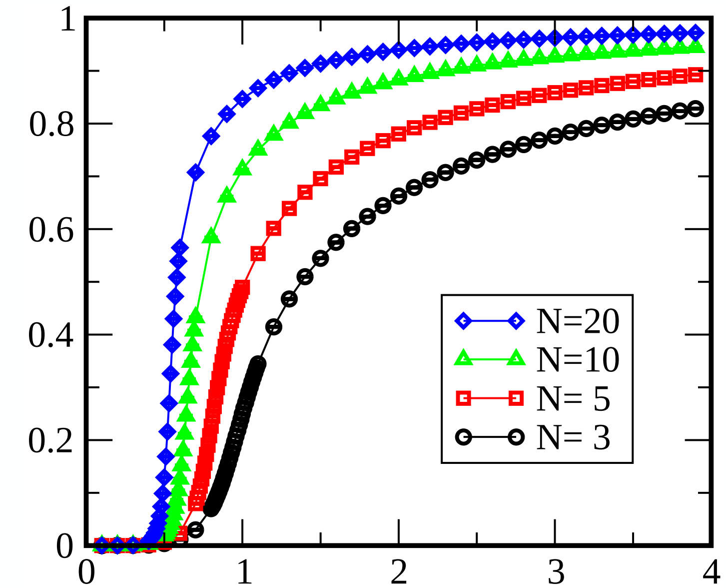
Main questions

◆Question 1 : Z_N (phase) transition for pbc

- 2nd-order phase transition expected in large-N
- it should be crossover for finite N since $\cancel{Z_N}$

$$|\langle P\rangle| \sim 0 \text{ for small } \beta \quad \xrightarrow{?} \quad |\langle P\rangle| \neq 0 \text{ for large } \beta$$

We will check it directly in numerical study

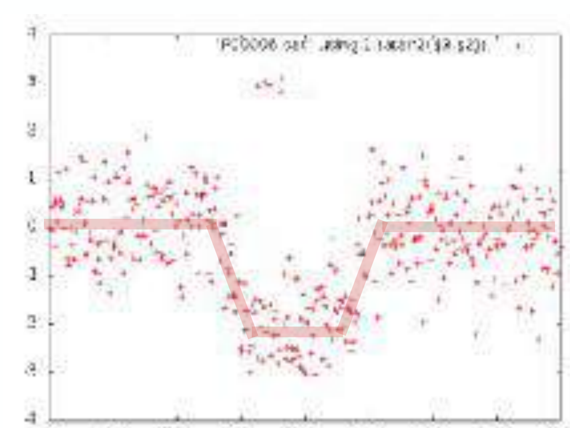


◆Question 2 : Continuity and fractional instantons for Z_N-tbc

- Fractional instantons yield transition between classical N-vacua
- makes Z_N stable, leading to volume indep. of vacuum structure

$$|\langle P\rangle| \sim 0 \text{ for small } \beta \quad \xrightarrow{?} \quad \text{still } |\langle P\rangle| \sim 0 \text{ for large } \beta$$

We will show quite suggestive results on fractional instantons and adiabatic continuity



Setup of lattice simulation

cf.) Berg,Luscher(81), Campostrini,et.al.(92),Alles,et.al.(00), Flynn,et.al.(15),Abe,et.al.(18)

- **Lattice formulation** $S = -N\beta \sum_{n,\mu} \left(\bar{z}_{n+\mu} \cdot z_n \lambda_{n,\mu} + \bar{z}_n \cdot z_{n+\mu} \bar{\lambda}_{n,\mu} - 2\right)$

  Vector field Φ is introduced: $\begin{aligned} \phi_{2j} &= \Re[z_{n,j}], \quad \phi_{2j+1} = \Im[z_{n,j}], \quad j = 0, \cdots, N-1 \\ \phi^R_\mu &= \Re[\lambda_\mu], \quad \phi^I_\mu = \Im[\lambda_{n,\mu}], \end{aligned}$

  ➡ $s_\phi = -N\beta\phi \cdot F_\phi = -N\beta |F_\phi| \cos\theta$ updated just by updating θ

  Over heat-bath algorithm is adopted to update this θ

- **Parameters and quantities**

  N_x = 40-400, N_τ = 8,12, β= 0.1-4.0, N = 3-20, $N_{\rm sweep}$ = 200000,400000

  - Expectation values of Polyakov loop and its susceptibility
  - Thermal entropy $s = \beta(N\tau)^2(\langle T_{xx}\rangle - \langle T_{\tau\tau}\rangle)$

(1)Z_N transition(pbc) (2)Z_N continuity(tbc) (3)Thermal entropy

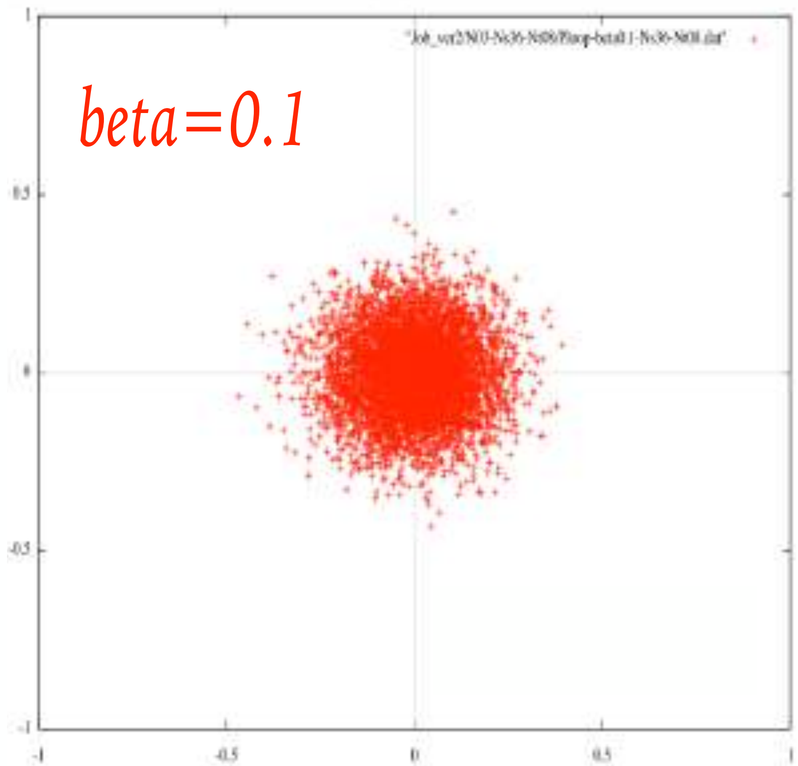
Polyakov-loop of CP^{N-1} models on R x S^1 with pbc.

N=3,5,10,20 (Nx,Nt) = (200, 8) Nsweep = 200,000

Distribution plot of P-loop

Im[P]

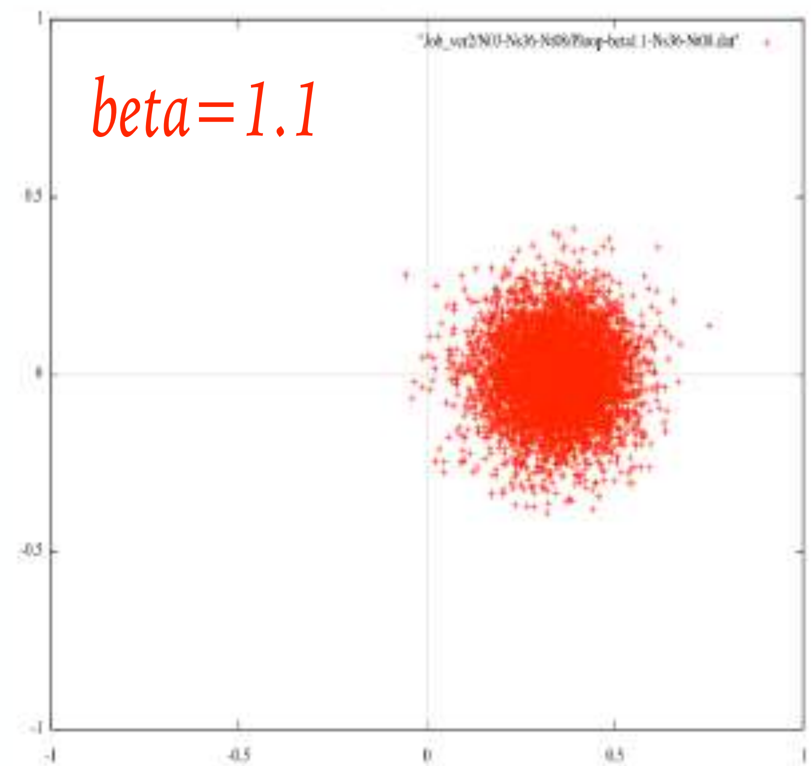
N=3



Re[P]

$|\langle P \rangle| \sim 0$

N=3



$|\langle P \rangle| \neq 0$

Low-β : around the origin

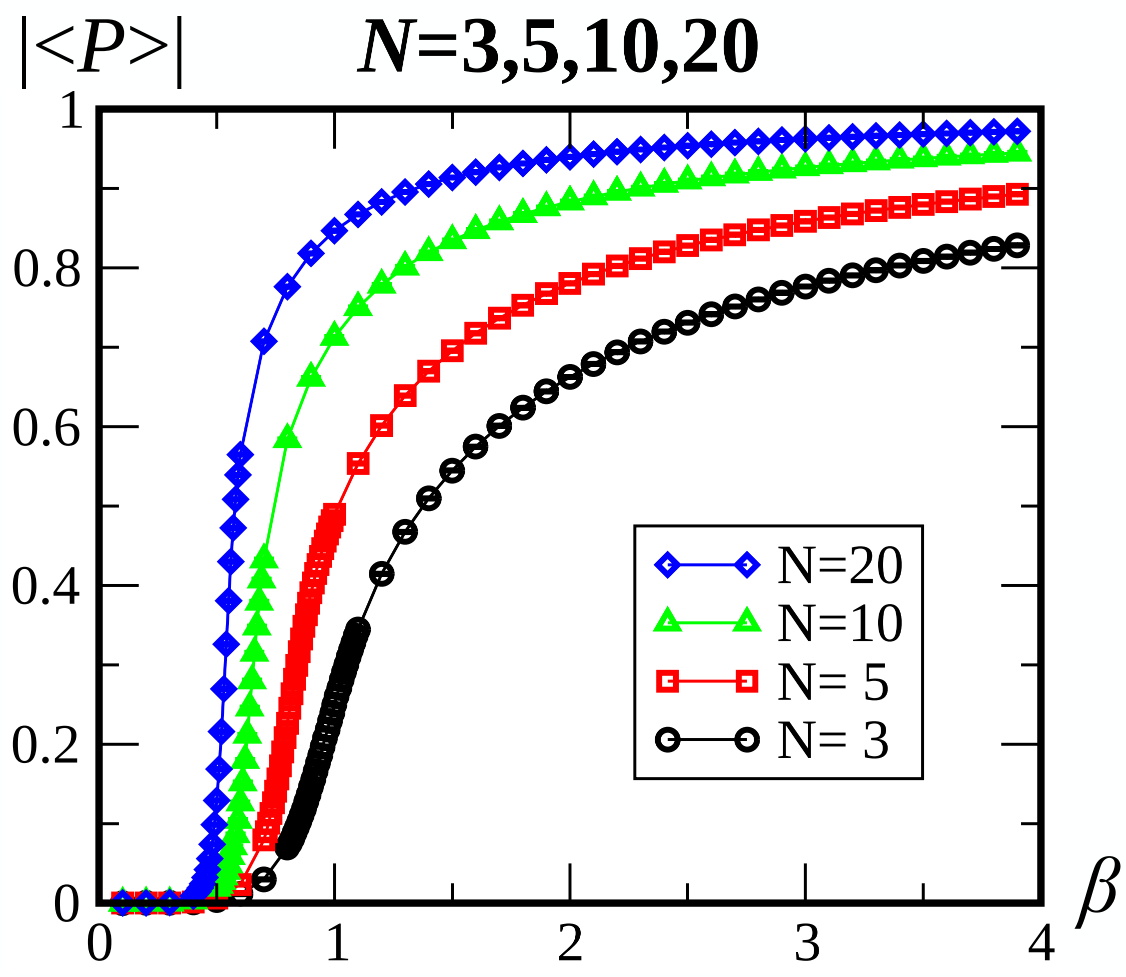
→ approximate Z_N symmetry

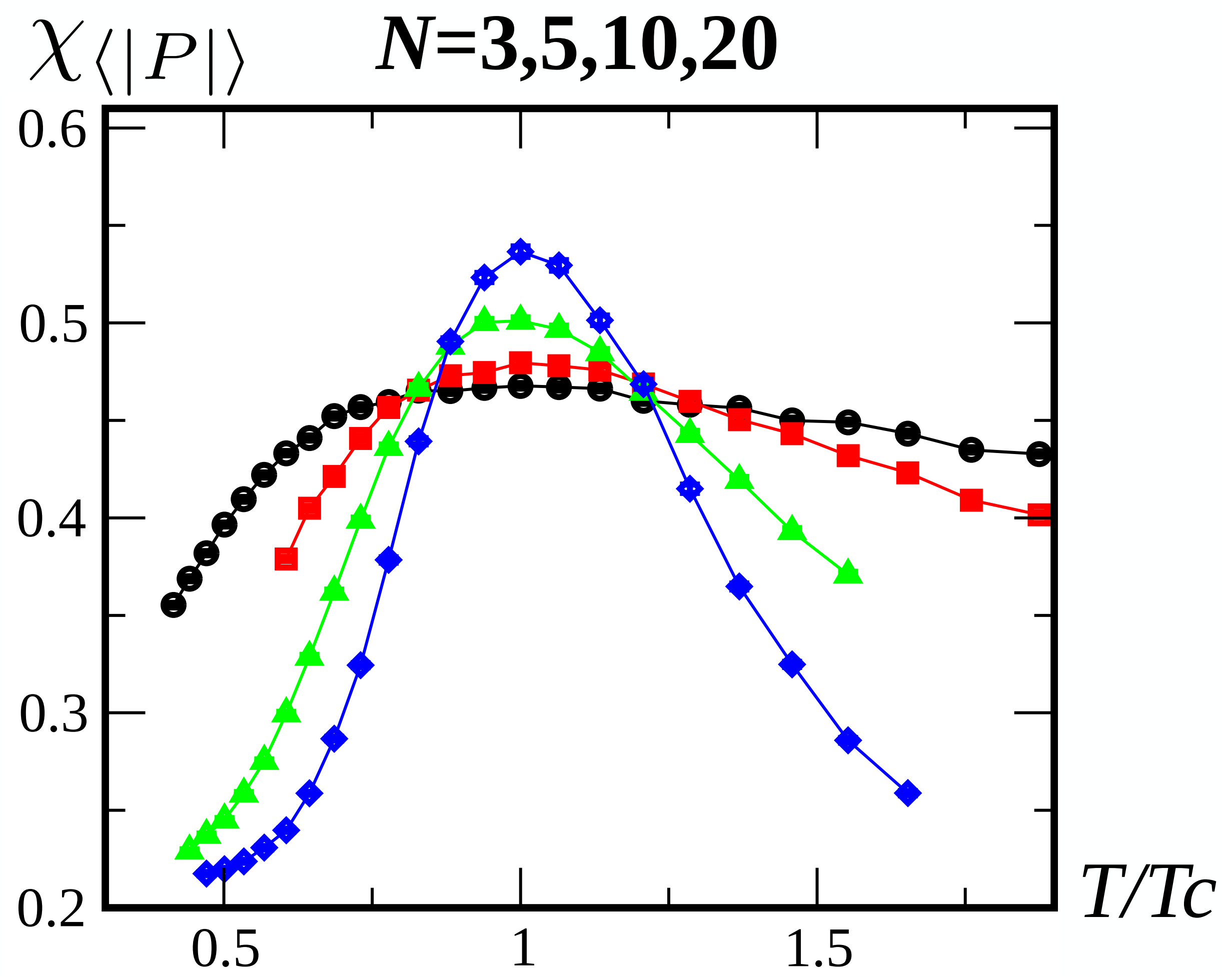
High-β : moves to one of Z_N vacua

→ Z_N breaking transition

Note that Z_N symmetry is not exact for PBC

VEV of Polyakov loop |<P>|

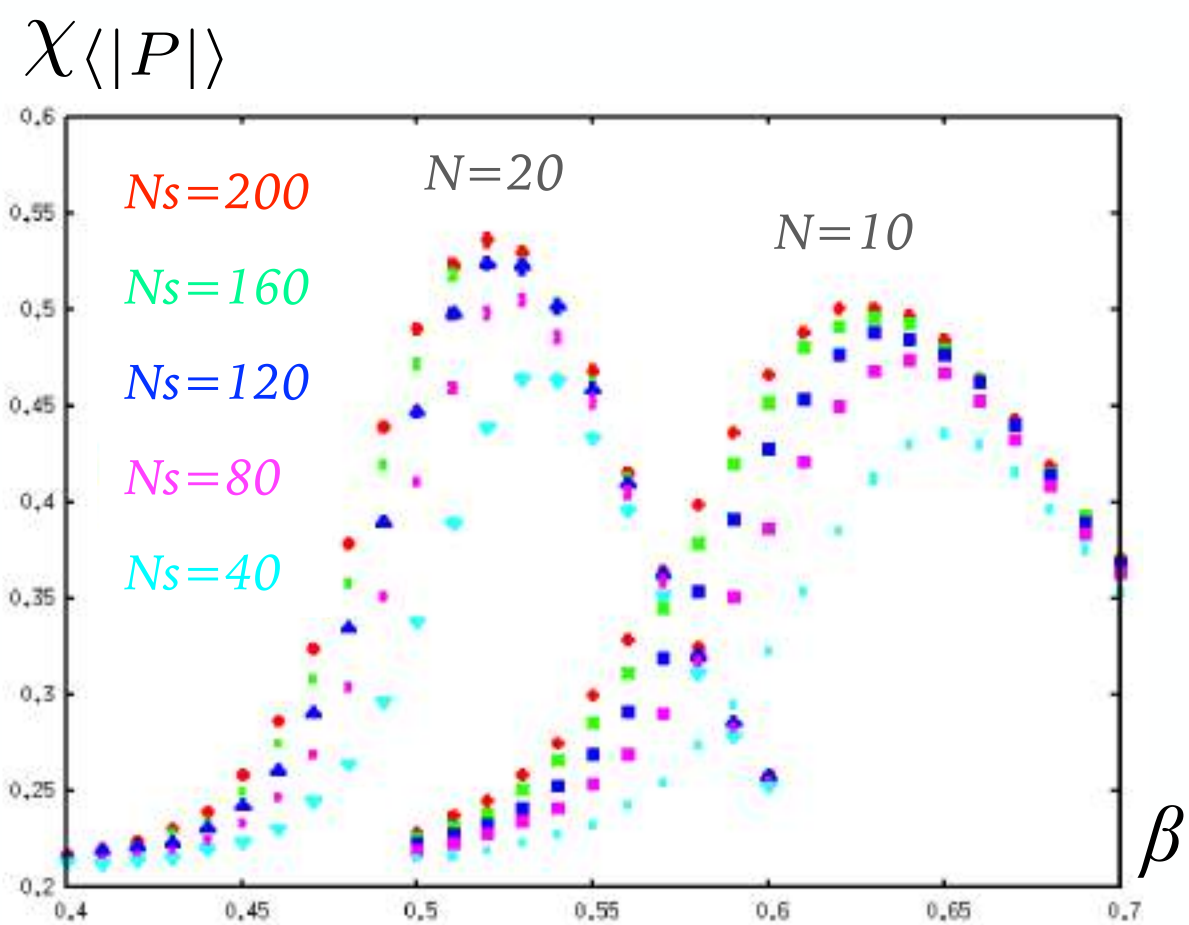


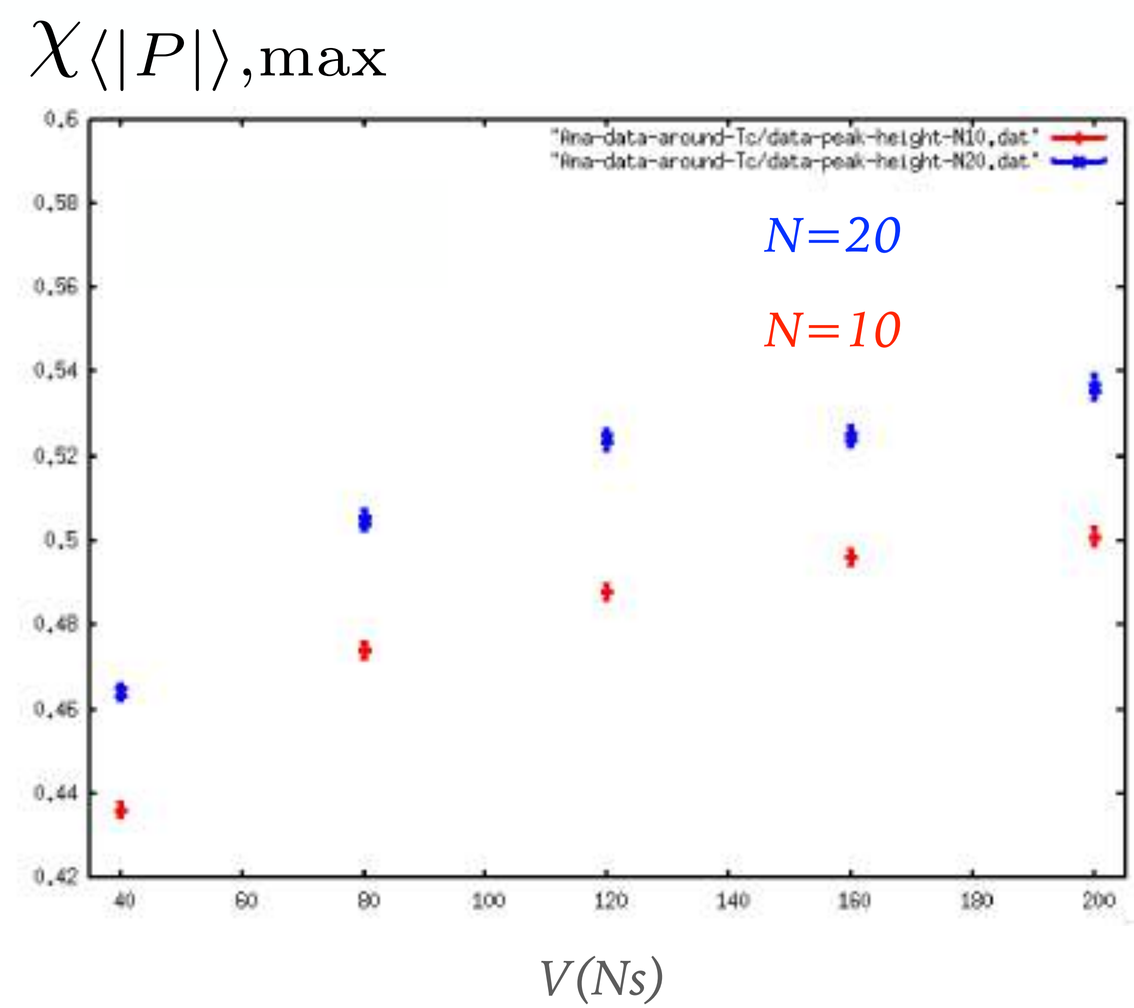


- |<P>| ~ 0 at low β, then |<P>| undergoes crossover-like transition
- Peak of Polyakov-loop susceptibility χ gets sharper with N

Crossover transition for finite N is checked,
which would be 2nd-order phase transition for large N limit

Volume dependence of χ-peak





$$\chi_{\max} = c + aV^p$$ **p=1 : 1st, 0<p<1: 2nd or crossover**

Fukugita,et.al.(90)

- Volume dependence of the peak is not linear → not 1st-order
- χ for N=20 is larger than that for N=10 → 2nd-order in large N?

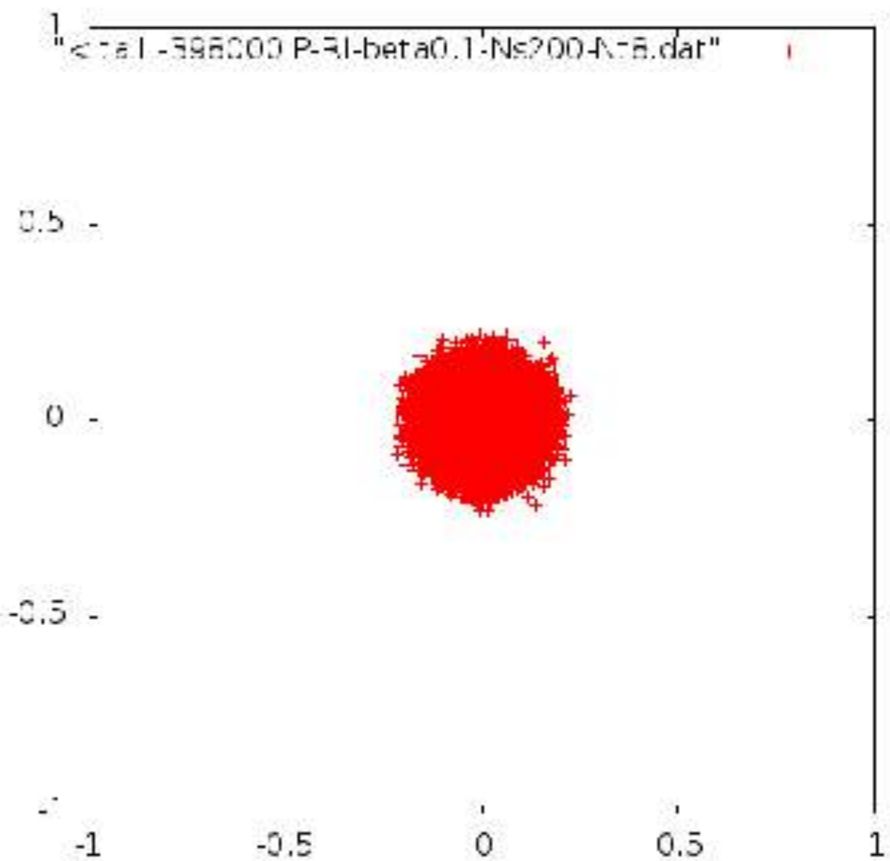
it supports crossover transition for finite N (2nd-order in large-N)

Polyakov loop of CP^{N-1} models on R x S^1 with Z_N tbc.

$N=3,5,10,20$ $(Nx,Nt) = (200, 8), (400,12)$ Nsweep=200000, 400000

Distribution plot of P-loop

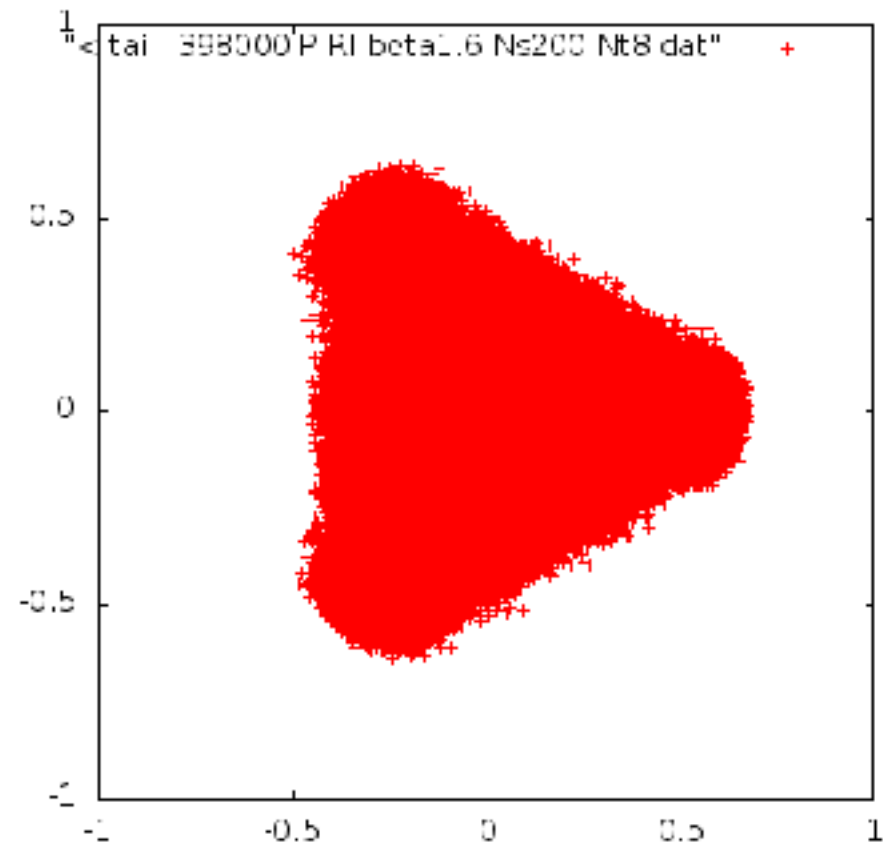
Im[P] $\boldsymbol{N=3, \beta=0.1}$



$|\langle \boldsymbol{P} \rangle| \sim \mathbf{0}$

$\boldsymbol{N=3, \beta=1.6}$

Re[P]



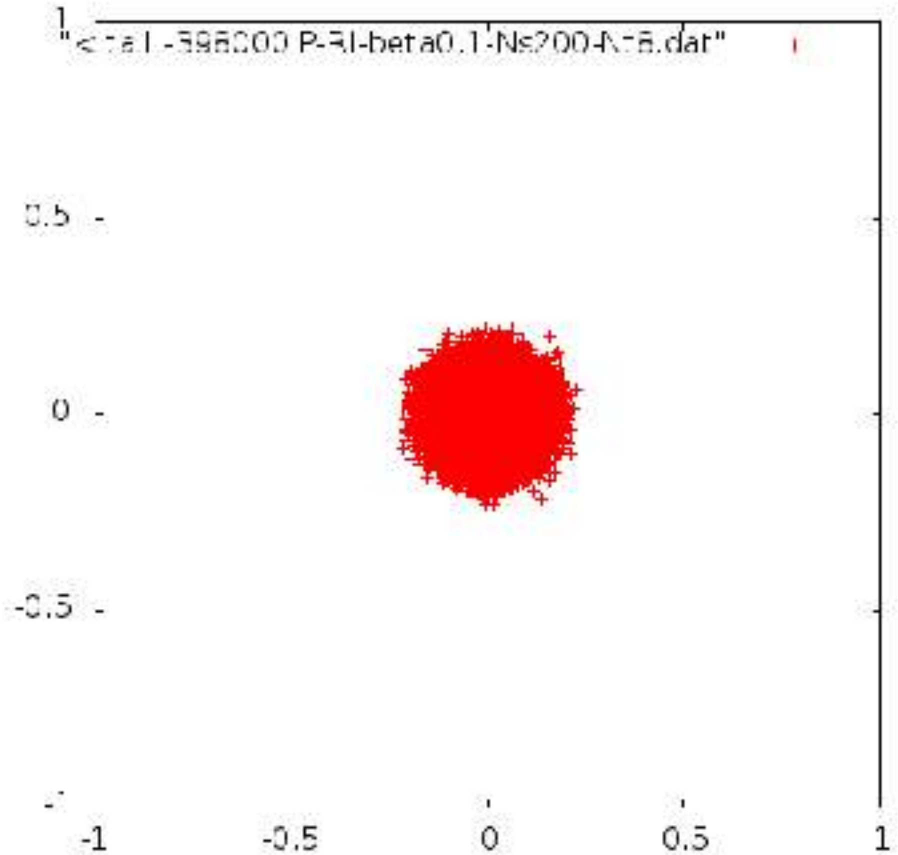
$|\langle \boldsymbol{P} \rangle| \sim \mathbf{0}$

Low-β : around the origin → Z_N symmetry at the action level

Intermediate-β : Transition between N vacua → quantum Z_N symmetry

Distribution plot of P-loop

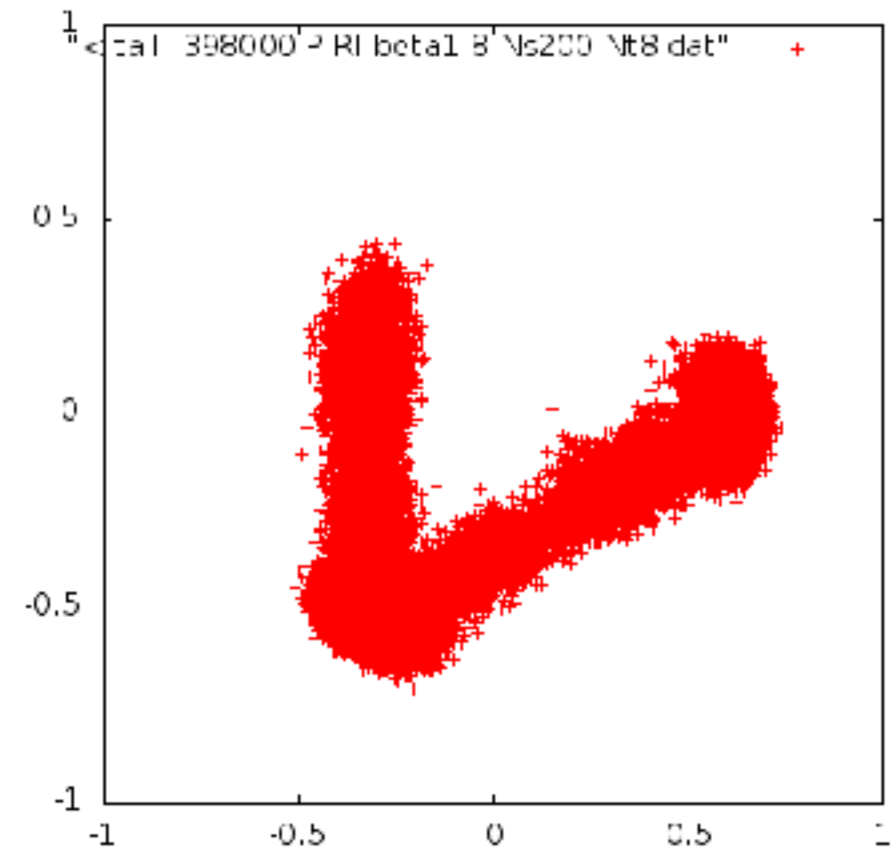
Im[P] N=3, β=0.1



Re[P]

$|\langle P \rangle| \sim 0$

N=3, β=1.8



$|\langle P \rangle| \neq 0$

Low-β : around the origin →

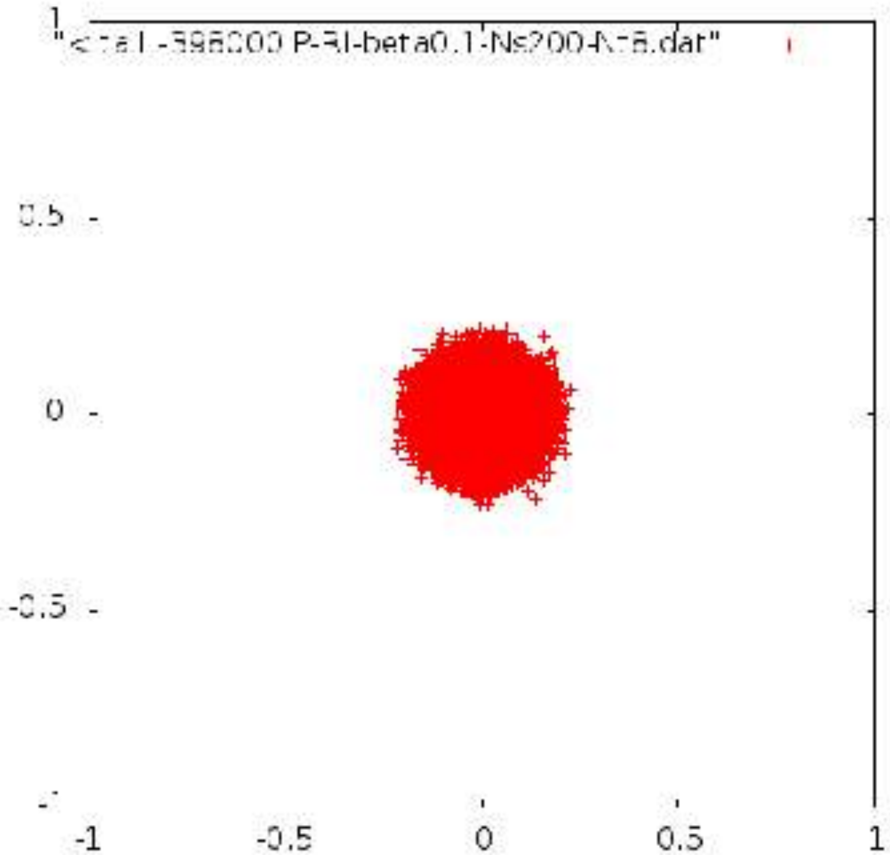
Z_N symmetry at the action level

High-β : One of Z_N vacua selected

→ SSB of Z_N symmetry….?

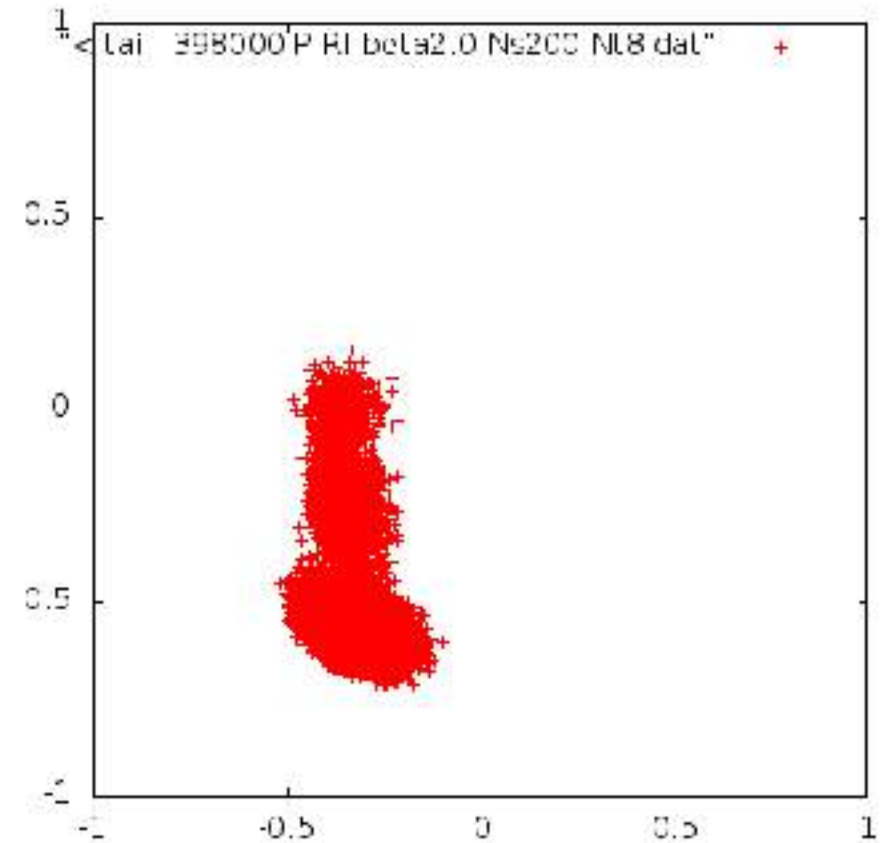
Distribution plot of P-loop

Im[P] **N=3, β=0.1**



Re[P]

N=3, β=2.0



$|\langle P\rangle| \sim 0$

$|\langle P\rangle| \neq 0$

Low-β：around the origin →

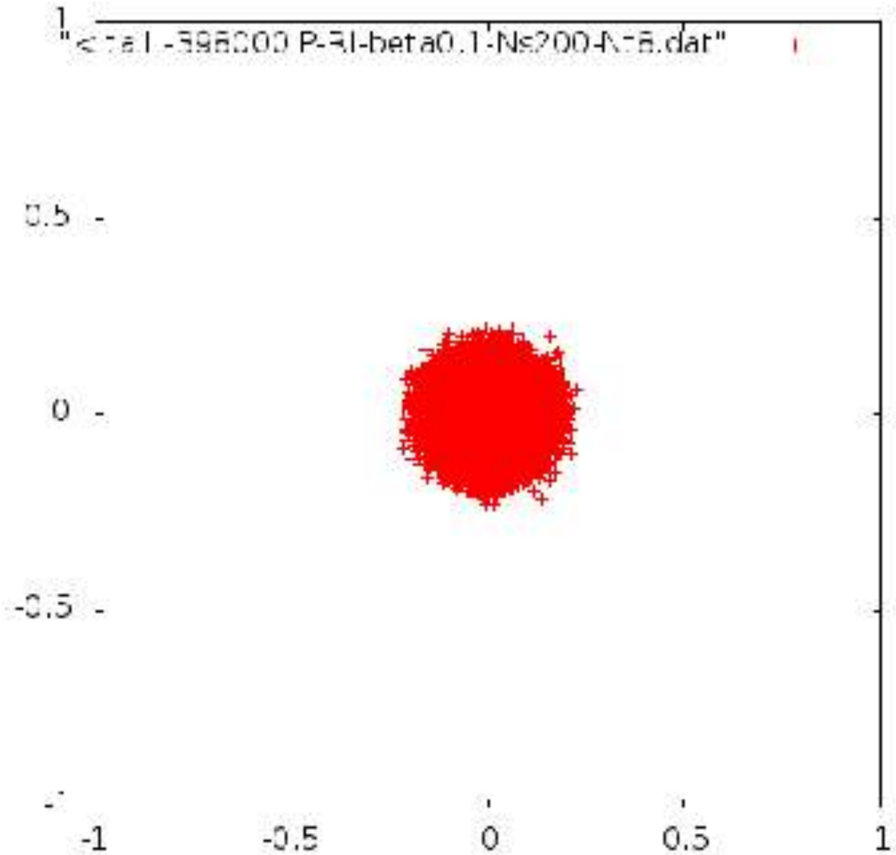
Z_N symmetry at the action level

High-β：One of Z_N vacua selected

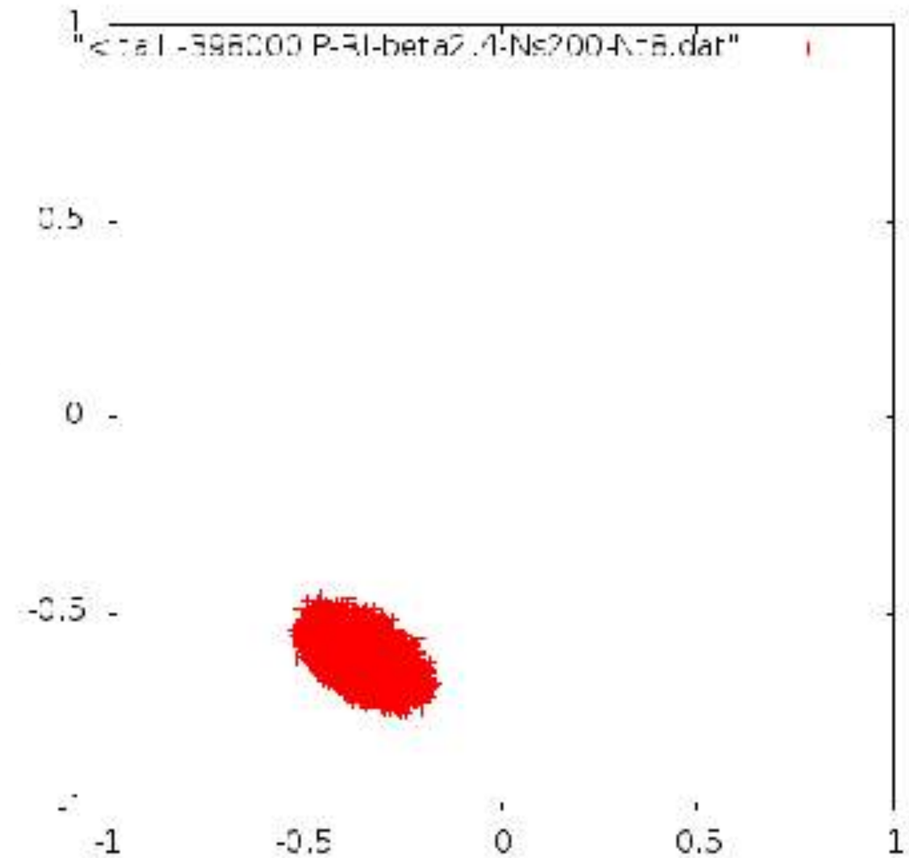
→ SSB of Z_N symmetry….?

Distribution plot of P-loop

Im[P] N=3, β=0.1



N=3, β=2.4



Re[P]

$|\langle P \rangle| \sim 0$

$|\langle P \rangle| \neq 0$

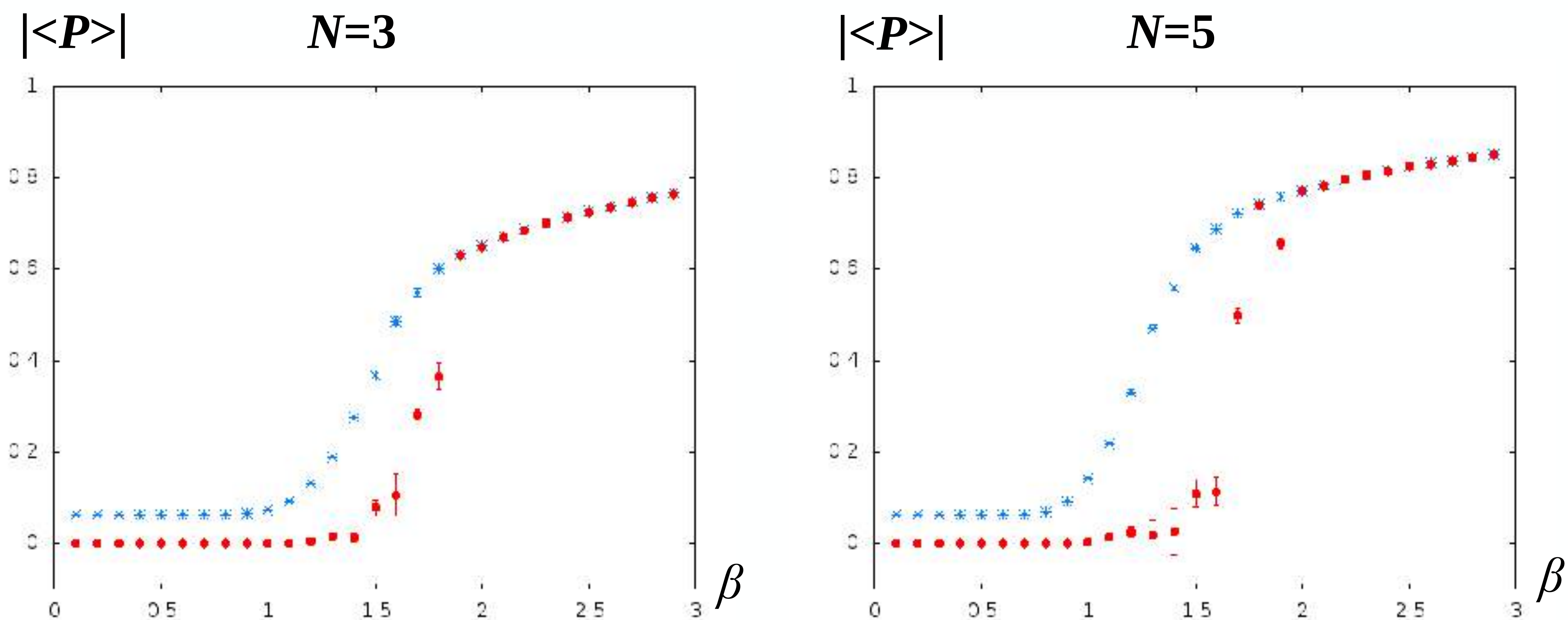
Low-β：around the origin →

Z_N symmetry at the action level

High-β：One of Z_N vacua selected

→ SSB of Z_N symmetry….?

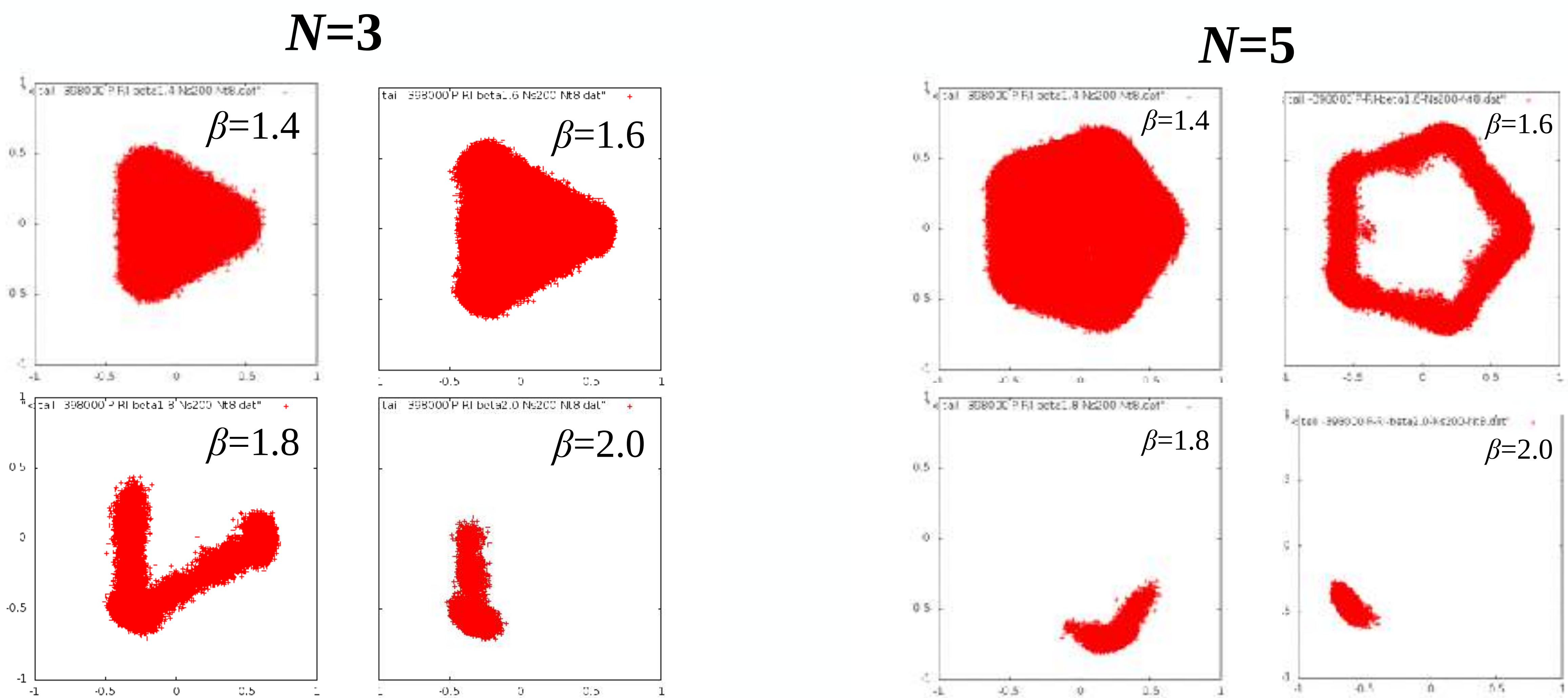
VEV of Polyakov loop |<P>|



- Low β → |<P>|= 0 : distribution around origin
- Mid β → |<P>| highly fluctuates : distribution forms polygons
- High β → suddenly gets |<P>|≠ 0 : but more stat. can form polygon

This peculiar P-loop could imply something special (Z_N stability?).
We still need larger volume or more statistics to judge continuity.

VEV of Polyakov loop |<P>|



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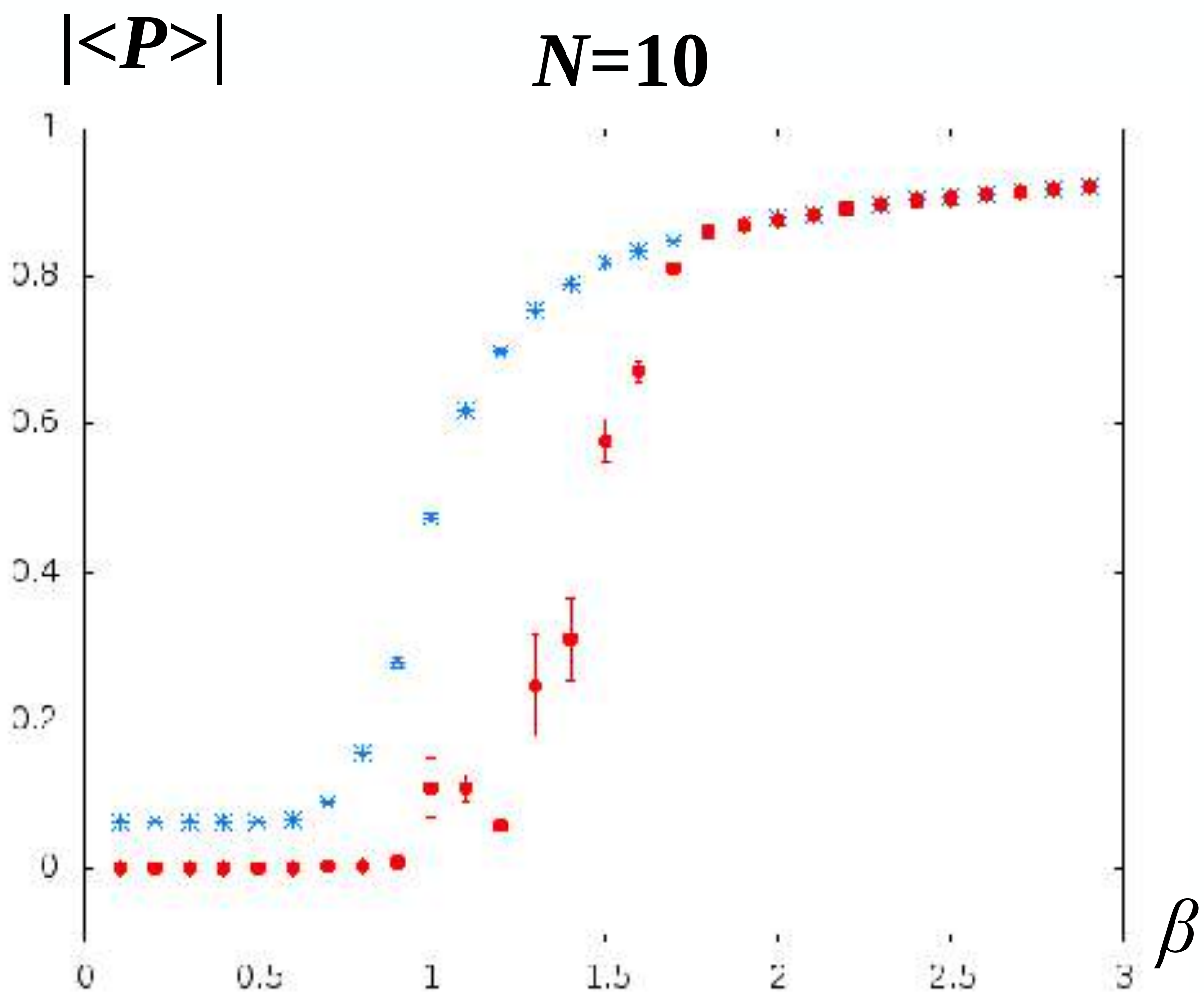

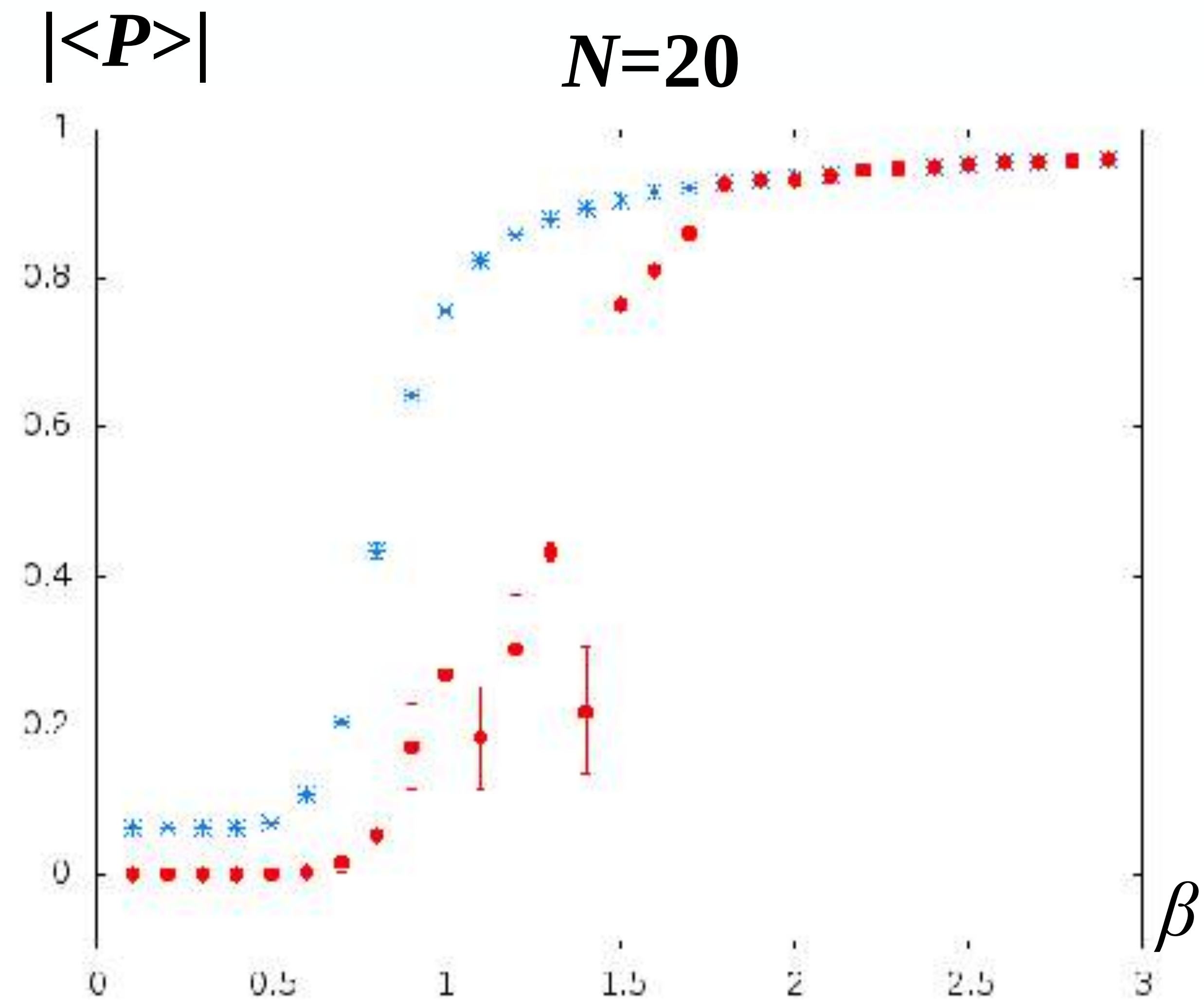

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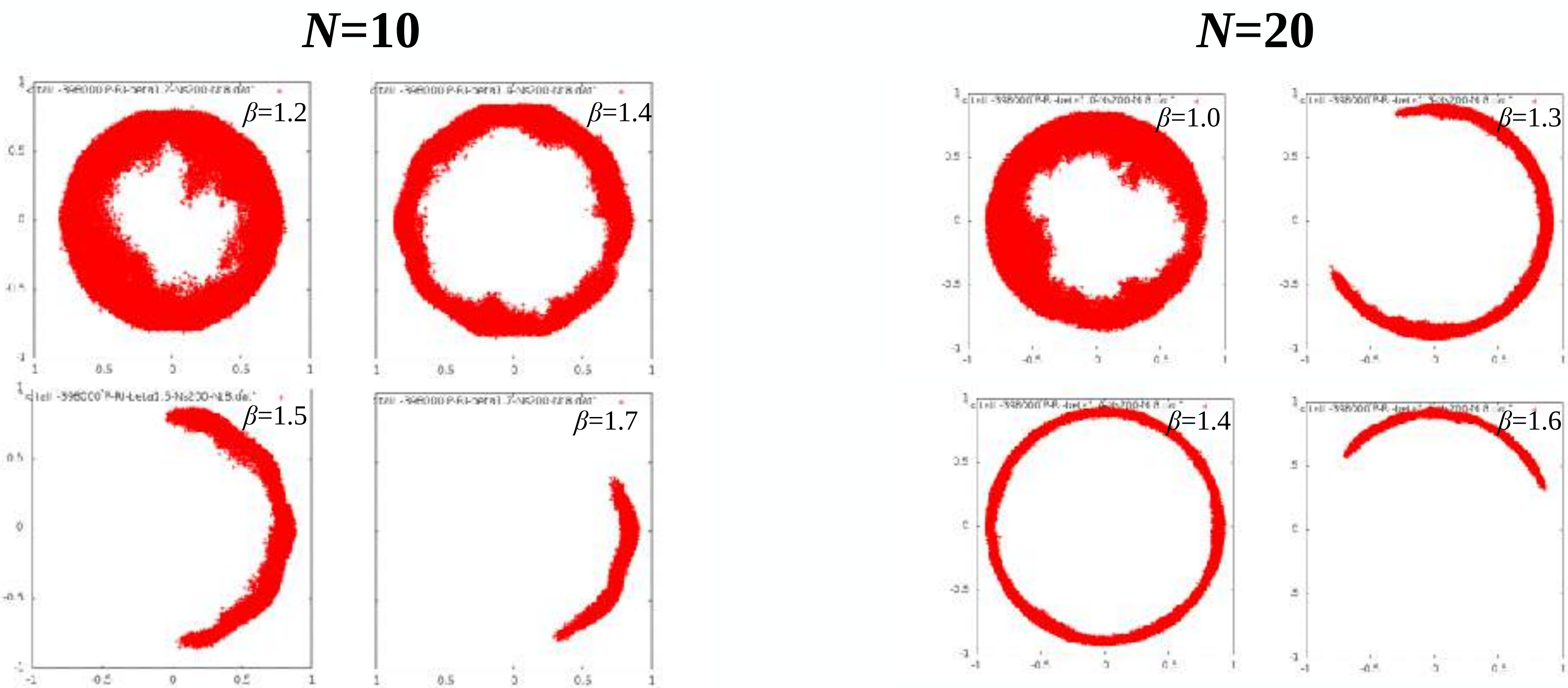
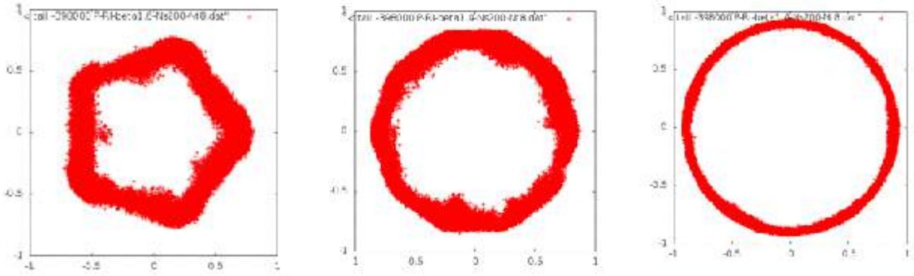
VEV of Polyakov loop |<P>|



- Low β → |<P>|= 0 : distribution around origin
- Mid β → |<P>| highly fluctuates : distribution forms polygons
- High β → suddenly gets |<P>|≠ 0 : but more stat. can form polygon

This peculiar P-loop could imply something special (Z_N stability?).
We still need larger volume or more statistics to judge continuity.

Polygon-shaped distributions of Polyakov loop (|<P>|~0) appear more often with more statistics



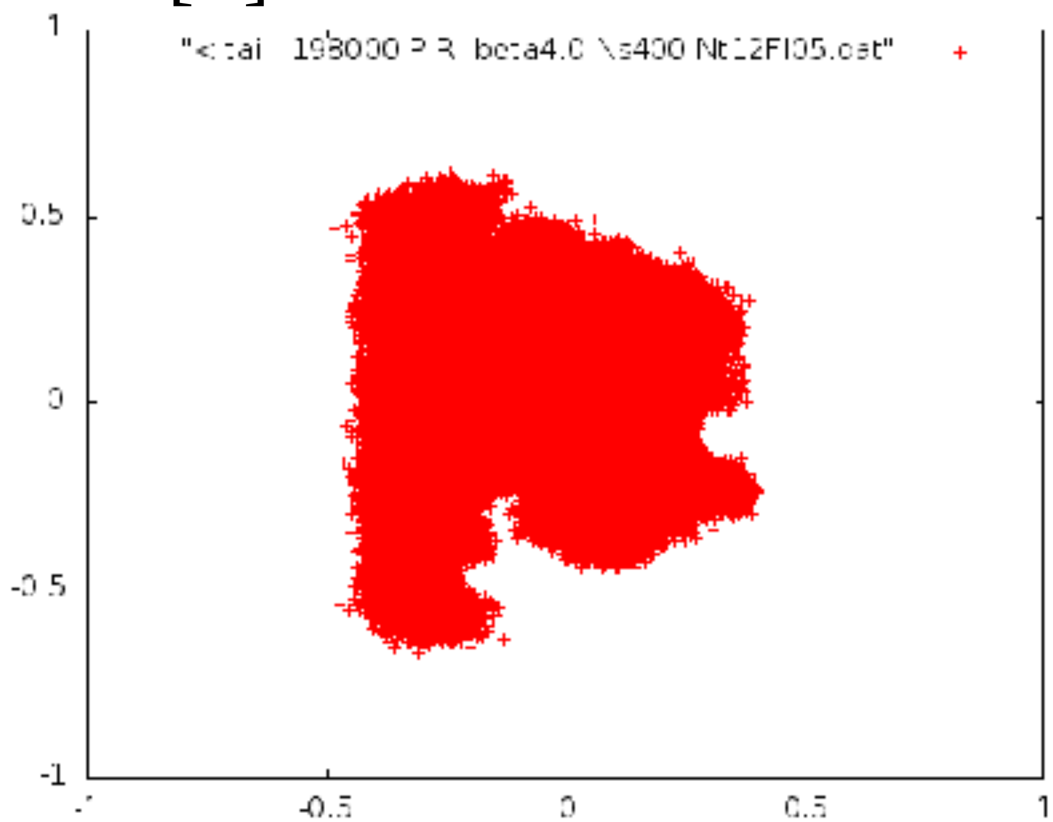
It may indicate Z_N stability (continuity)….

Furthermore,

Distribution plot of P-loop (very high β, large volume)

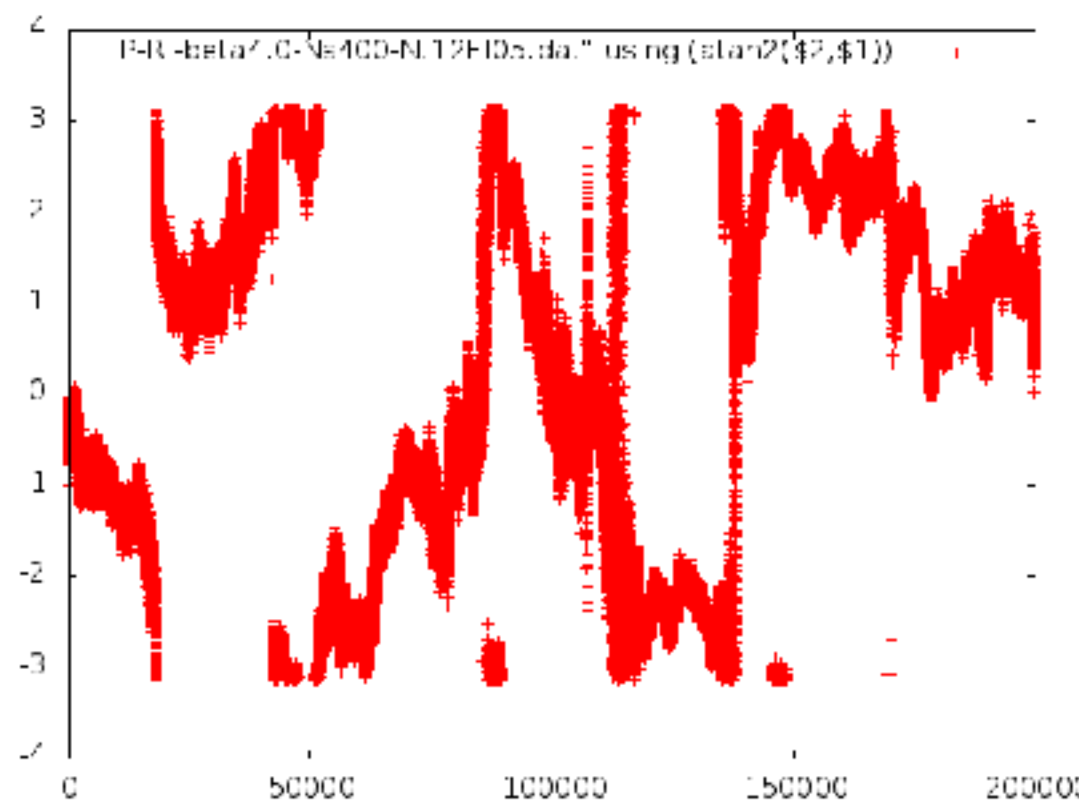
Independent configurations for very high β (β=4.0) with large volume include a quantum Z_N symmetric case as below !

Im[P] **N=3, β=4.0, (400×12)**



Re[P]

Hysteresis of arg[P]



|<P>| ~ small

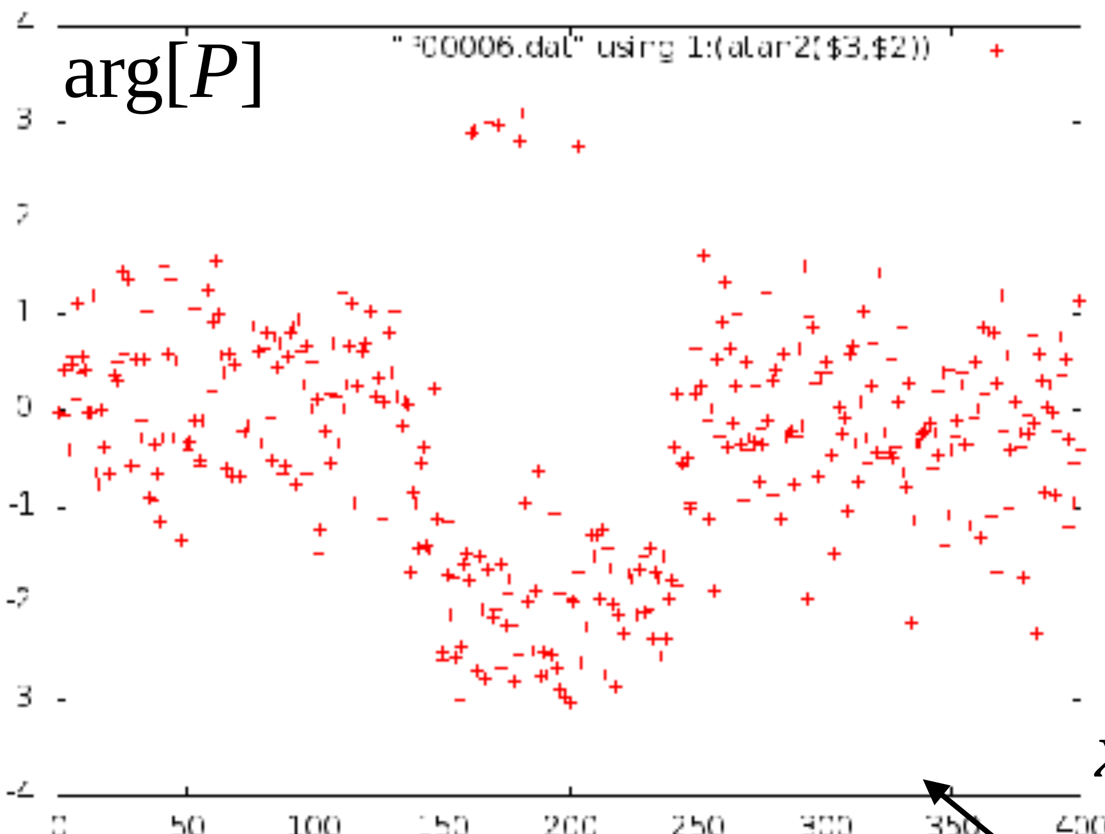
Any of Z_N vacua is not selected

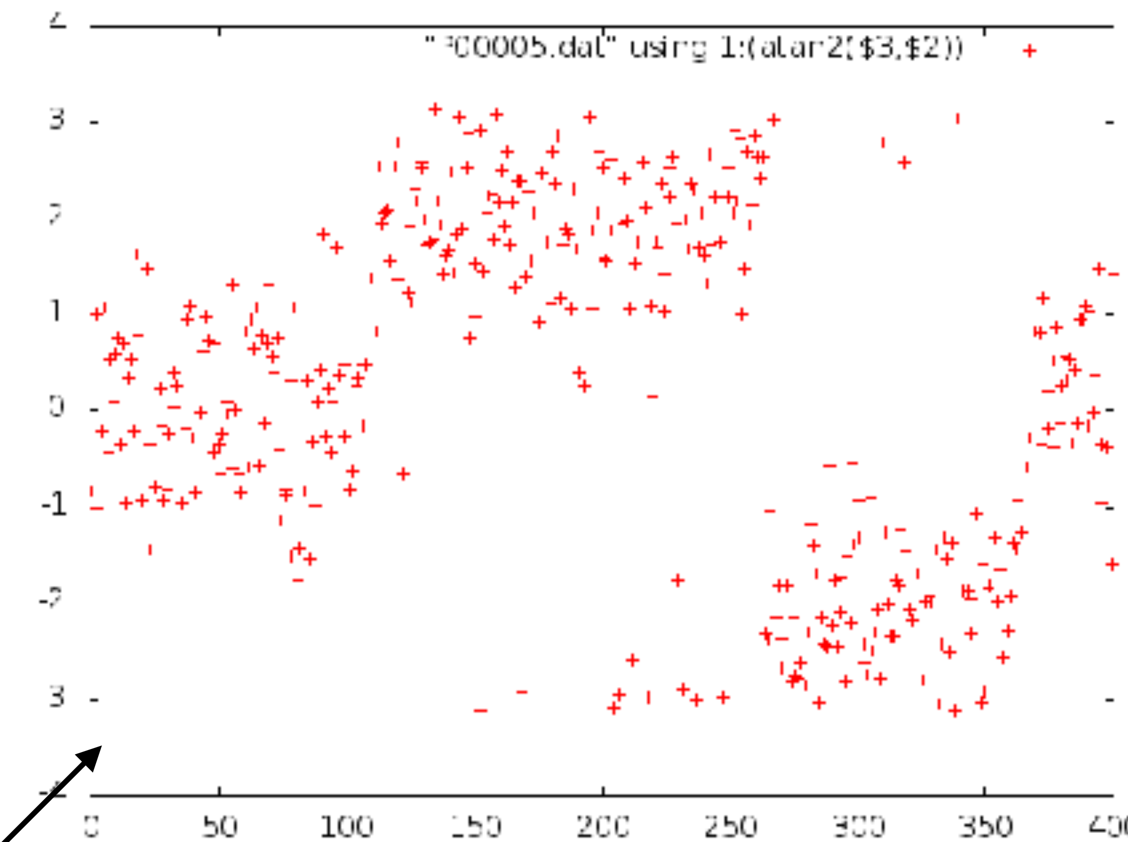
Very high-β : quantum Z_N symmetric case found with certain probability

it seems we need larger volume or more statistics for Z_N continuity….

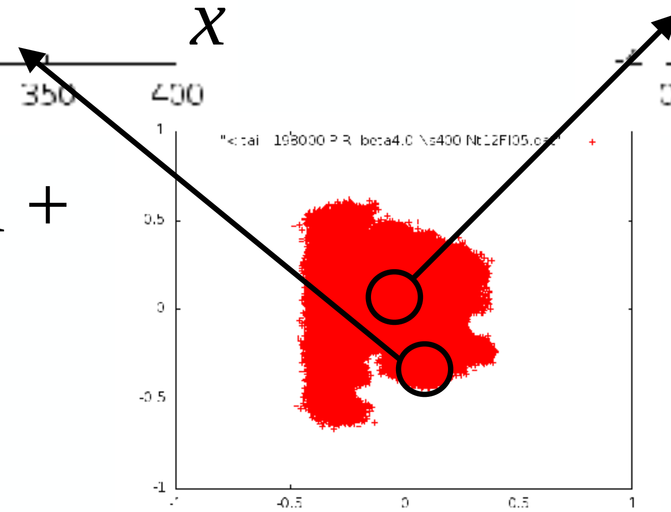
Fractional instantons

Pick up two of configurations and look into the x-dependence of arg[P]





1/3 fractional antiintanton +
1/3 fractional instanton
= **bion**

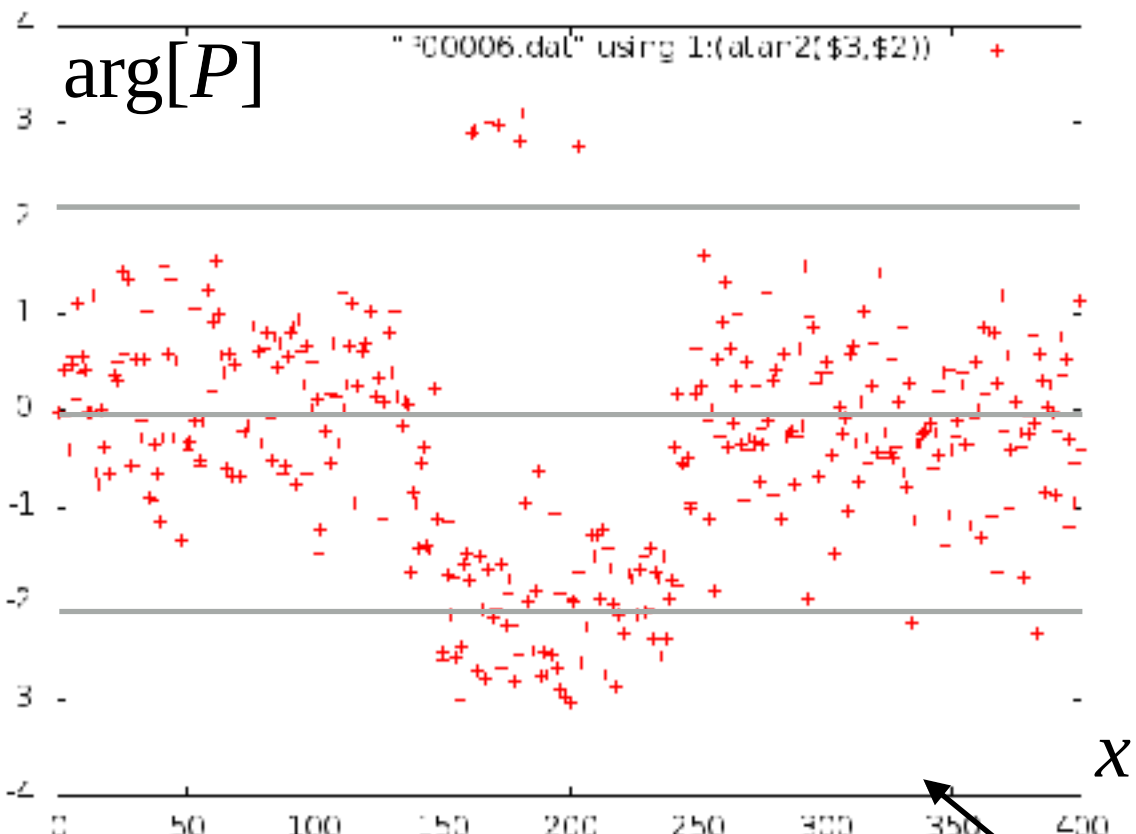


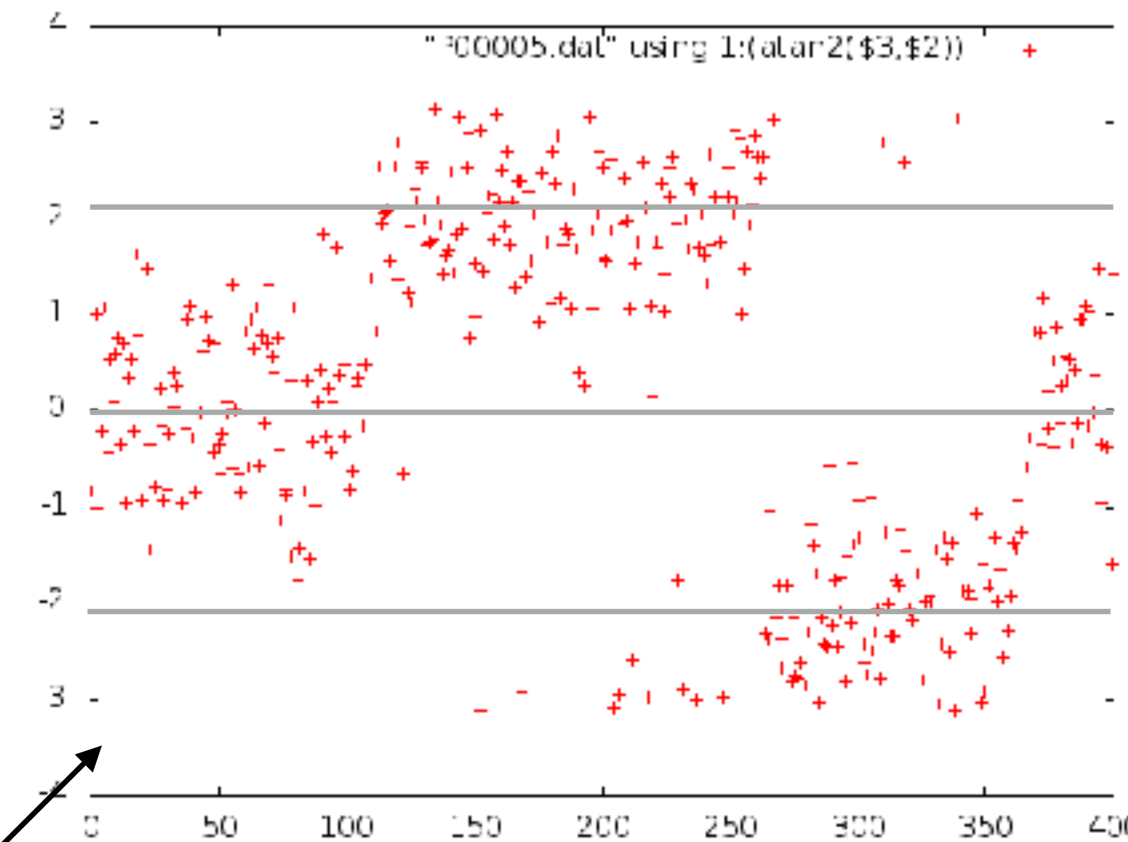
3 × 1/3 fractional intantons
= **instanton**

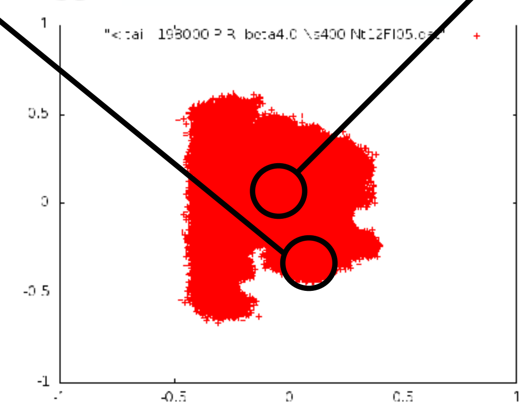
implies fractional instantons cause transition between classical vacua at high β, which lead to quantum Z_N symmetry and could yield adiabatic continuity

Fractional instantons

Pick up two of configurations and look into the x-dependence of arg[P]







1/3 fractional antiintanton +
1/3 fractional instanton
= **bion**

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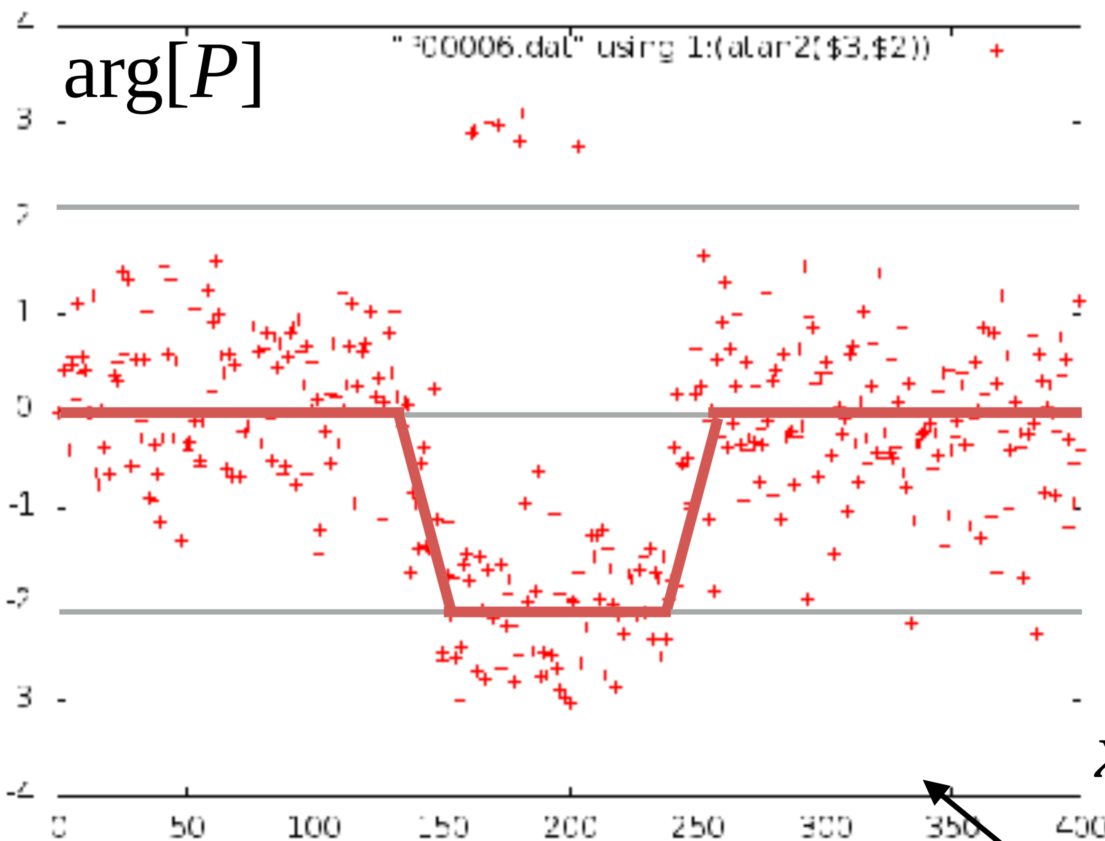
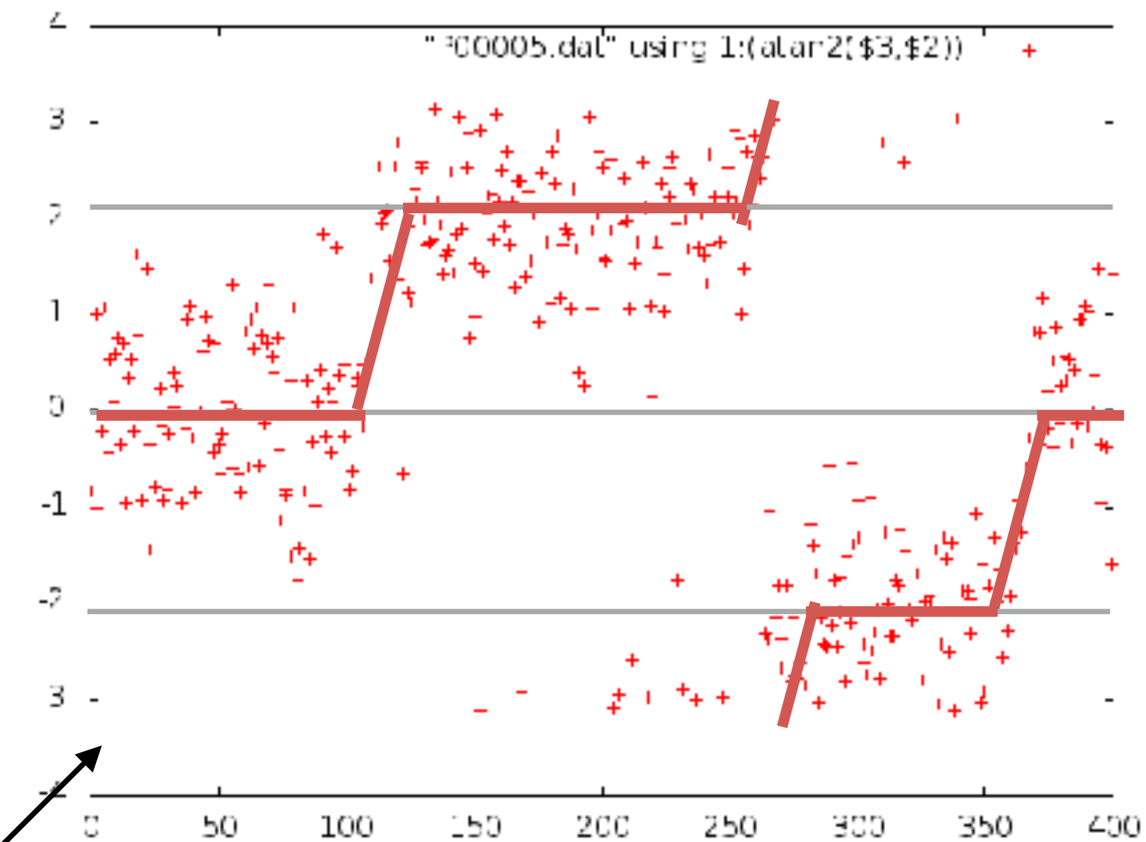
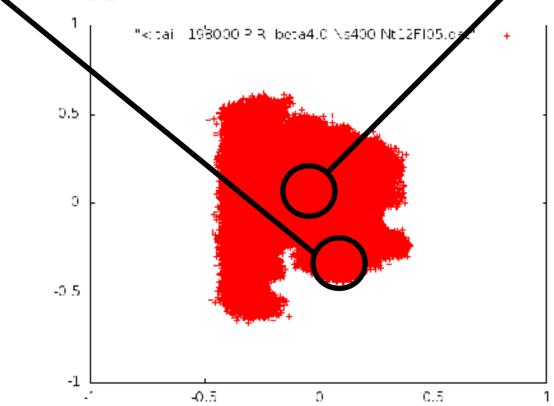
Fractional instantons

Pick up two of configurations and look into the x-dependence of arg[P]







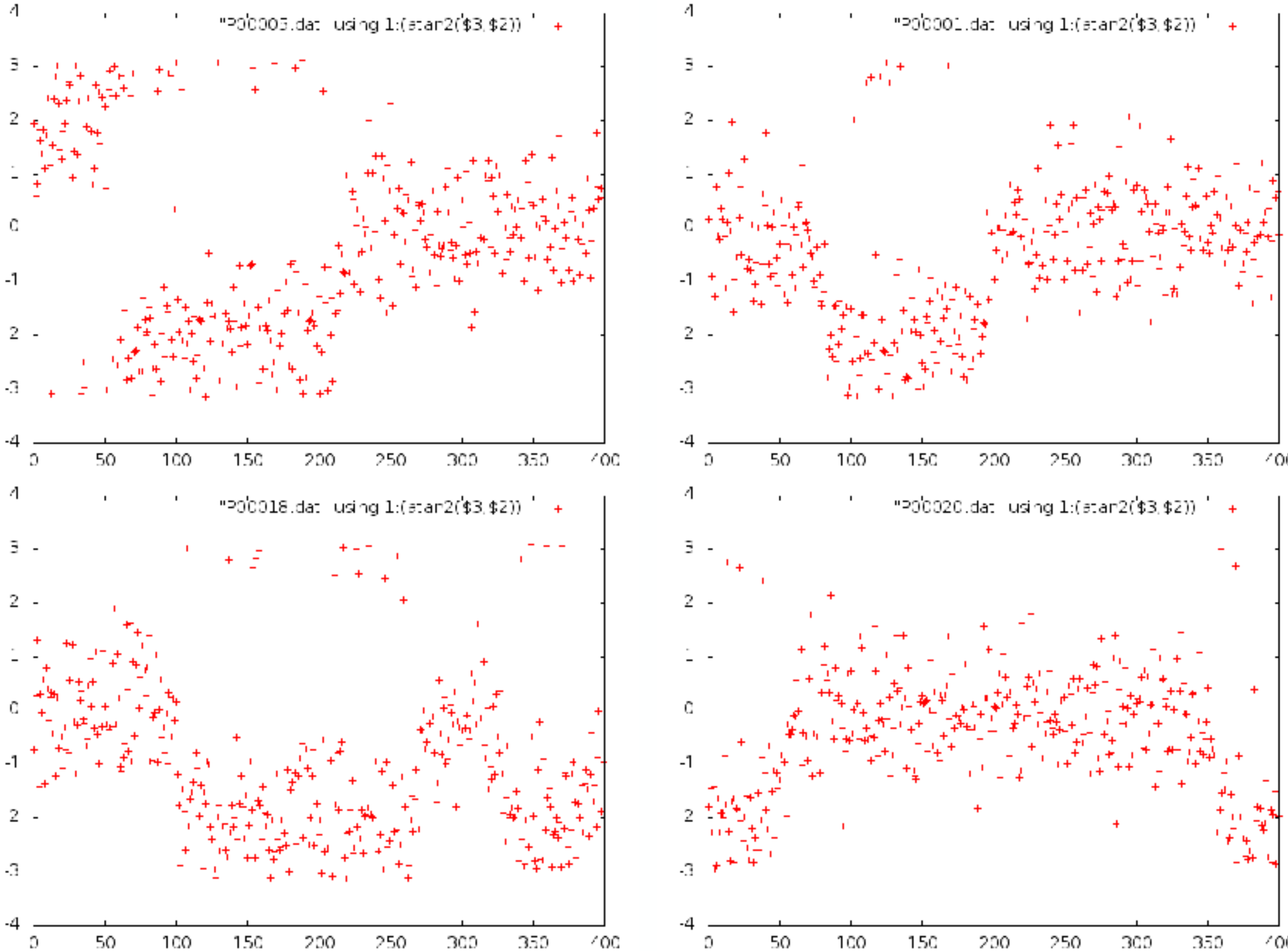
1/3 fractional antiintanton +
1/3 fractional instanton
= **bion**

3 × 1/3 fractional intantons
= **instanton**

implies fractional instantons cause transition between classical vacua at high β, which lead to quantum Z_N symmetry and could yield adiabatic continuity

＊we are on the way of calculating topological charge density directly.

Fractional instantons



Summary

- Lattice simulation of CP^{N-1} model on $R \times S^1$
- Z_N crossover transition is confirmed for pbc
- Thermal entropy agrees with the prediction for pbc
- Characteristic β dependence of P-loop for tbc, which inspires more study on adiabatic continuity
- A pivotal role of fractional instantons is implied for tbc