Symmetries in supersymmetric gauge theory on the graph

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Based on
N. Sakai and KO, PTEP **2019** 043B01,
and work in progress with S. Kamata, S. Matsuura and T. Misumi

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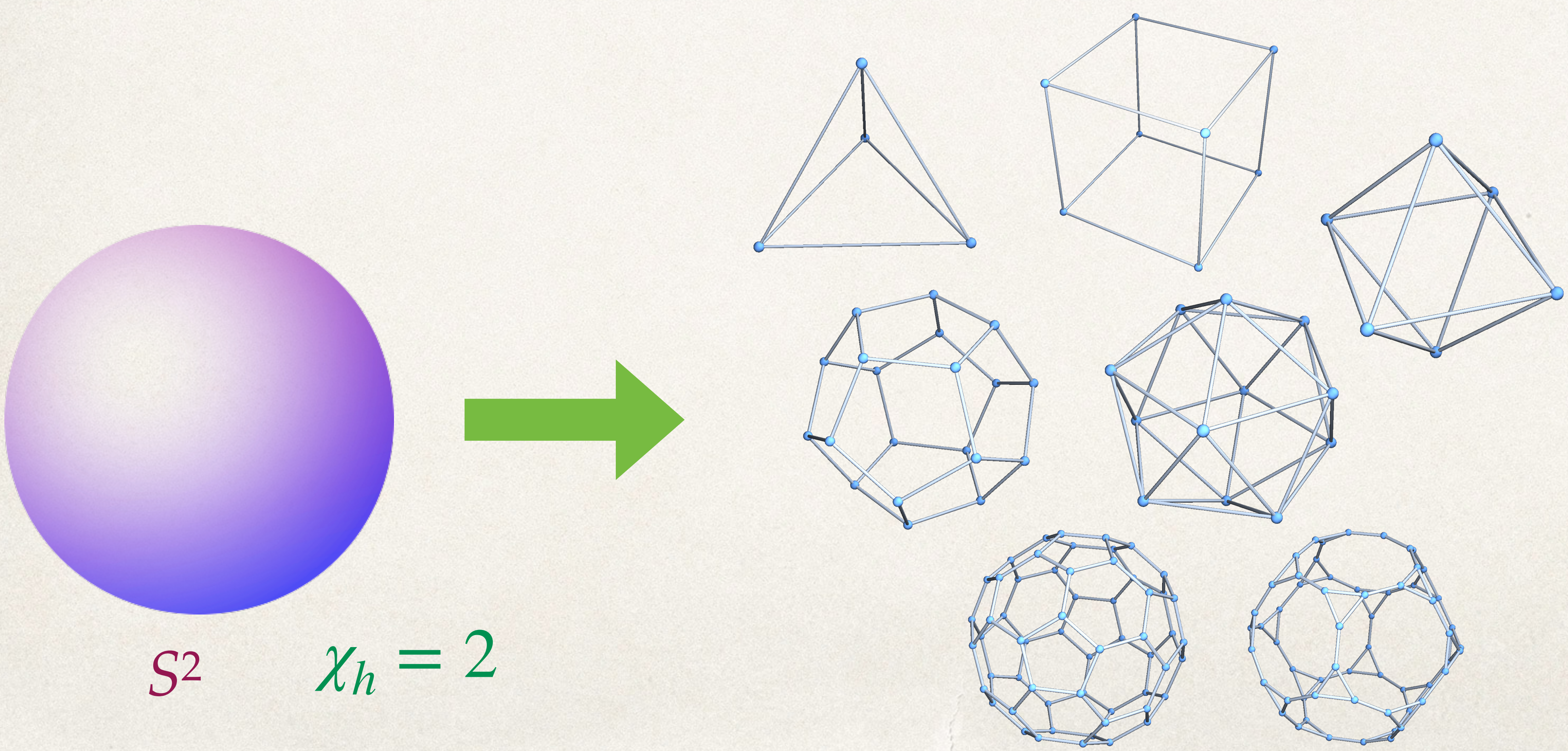
Introduction

2d supersymmetric (topological) gauge theory can be well formulated on generic graphs (discretized Riemann surface or polyhedra)
$\Rightarrow$ a generalization of the supersymmetric lattice gauge theory (the so-called Sugino model)



S^2 $\quad \chi_h = 2$

Simplicial complexes (graph) with the same Euler characteristics

$\chi_\Gamma = 2$

Introduction

We would like to consider properties (symmetries) of the discretized gauge theory on the 2d graph

Question:

How much can we discuss symmetries on the graph in parallel with the continuous field theory?

- Supersymmetries
- Global symmetries
- Index theorem, heat kernel, zero modes
- BRST symmetries, etc.

SUSY on curved Riemann surface

4d N=1 (4 supercharges)

$A_M,\ D; \quad \psi_\alpha,\ \bar{\psi}^{\dot{\alpha}}$

Riemann surface with genus h

dimensional reduction on $\Sigma_h \times T^2$

$A_\mu,\ \Phi = A_3 + iA_4,\ \bar{\Phi} = A_3 - iA_4,\ D; \quad \psi_\alpha,\ \bar{\psi}^{\dot{\alpha}}$

turn on a background R-gauge field

Preserves 2 supercharges at least

$$\nabla^R_\mu \xi \equiv \nabla_\mu \xi + i\mathscr{A}^R_\mu \xi = 0$$
$$\nabla^R_\mu \bar{\xi} \equiv \nabla_\mu \bar{\xi} - i\mathscr{A}^R_\mu \bar{\xi} = 0$$

Killing eq.

SUSY on curved Riemann surface

original fields	helicity	R-charge	redefined fields	
$\Phi,\ \bar{\Phi}$	0	0	0-form	$\Phi,\ \bar{\Phi}$
A_μ	± 1	0	1-form	$A = A_\mu dx^\mu$ volume form
D	0	0	2-form	$Y \equiv D\omega - F$ field strength
$\psi_1,\ \bar{\psi}_{\dot{1}}$	$\pm 1/2$	$\pm 1/2$	1-form	$\lambda = \lambda_\mu dx^\mu$
$\psi_2,\ \bar{\psi}_{\dot{2}}$	$\pm 1/2$	$\mp 1/2$	0-form 2-form	η $\chi = \frac{1}{2}\chi_{\mu\nu} dx^\mu \wedge dx^\nu$

as the same as the topological twist

Isometries and supercharges

- 4 supercharges are decomposed into:

  $$Q, \quad \cancel{Q_\mu}, \quad Q_{\mu\nu}\ (\tilde{Q}) \qquad \text{on generic curved Riemann surface}$$

  0-form 1-form 2-form

- 2 supercharges are nilpotent up to gauge transformation:

  $$Q^2 = \tilde{Q}^2 = \delta_g$$

- If there exist isometries, associated supercharges are preserved:

  $$Q_I^2 = \delta_g + \mathscr{L}_I \longleftarrow \text{Lie derivative}$$

  e.g. (squashed) sphere ⇒ 1 isometry ⇒ 3 supercharges

  torus ⇒ 2 isometries ⇒ 4 supercharges (2d N=(2,2) SUSY)

SUSY transformation

- We consider Abelian gauge theory only in this talk

- We can define SUSY transformations for one of the supercharges Q

$$
\begin{aligned}
&Q\phi = 0, \\
&Q\bar{\phi} = 2\eta, \quad Q\eta = 0 \\
&QA = \lambda, \qquad Q\lambda = -d\phi \\
&QY = 0, \qquad Q\chi = Y
\end{aligned}
$$

Note that $Q^2 = \delta_\phi$

- The action can be written in the Q-exact form

$$S = -\frac{1}{2g^2} Q \int \left[d\bar{\phi} \wedge {}^*\lambda + \chi \wedge {}^*(Y - 2F) \right]$$

SUSY action

- Bosonic part of the SUSY action:

$$S_b = \frac{1}{2g^2}\int \left[d\bar{\phi} \wedge {}^*d\phi - Y \wedge {}^*(Y - 2F)\right]$$

$$\Rightarrow \frac{1}{2g^2}\int \left[d\bar{\phi} \wedge {}^*d\phi + F \wedge {}^*F\right]$$

- Fermionic part of the SUSY action:

$$S_f = \frac{1}{2g^2}\int \Psi^T \wedge {}^*i\not{D}\Psi \equiv \frac{1}{2g^2}(\Psi, i\not{D}\Psi)$$

where

$$\Psi = \begin{pmatrix} \eta \\ \lambda \\ \chi \end{pmatrix}, \quad i\not{D} = \begin{pmatrix} 0 & -d^\dagger & 0 \\ d & 0 & d^\dagger \\ 0 & -d & 0 \end{pmatrix}, \quad d^\dagger \equiv -*d*$$

adjoint exterior derivative
(co-differential)

Another supercharge

- If we exchange a role between 0-forms (η) and 1-forms (χ), we can find another SUSY transformation $\tilde{Q}$

$$
\begin{aligned}
&\tilde{Q}(\phi\omega) = 0, \\
&\tilde{Q}(\bar{\phi}\omega) = 2\chi, \quad && \tilde{Q}\chi = 0 \\
&\tilde{Q}A = *\,\lambda, && Q\lambda = -\,d^{\dagger}(\phi\omega) \\
&\tilde{Q}Y = 0, && \tilde{Q}\eta = -\,*\,Y
\end{aligned}
$$

Again $\tilde{Q}^2 = \delta_\phi$

- The same Q-exact action also can be written in the $\tilde{Q}$-exact form

$$
S = \frac{1}{2g^2}\tilde{Q}\int \left[d\bar{\phi}\wedge\lambda + \eta(Y - 2F)\right]
$$

Q vs $\tilde{Q}$

- The action is invariant under Q and $\tilde{Q}$ (both Q and $\tilde{Q}$ exact) since the action can be written simply by
$$S = \frac{1}{4g^2}[Q, \tilde{Q}] \int \left[\bar{\phi}F + \eta\chi\right]$$
and
$$\{Q, \tilde{Q}\} = 0$$
- Thus 2 supercharges Q and $\tilde{Q}$ are preserved on the Riemann surface Σ_h

$U(1)_A$ current

- The action is invariant under the $U(1)_A$ rotation
  $\phi \to e^{2i\theta_A}\phi, \quad \bar{\phi} \to e^{-2i\theta_A}\bar{\phi}, \quad \eta \to e^{-i\theta_A}\eta, \quad \lambda \to e^{i\theta_A}\lambda, \quad \chi \to e^{-i\theta_A}\chi$

- Associated $U(1)_A$ current is given by
  $$J_A = \left(\phi d\bar{\phi} - d\phi\bar{\phi} + \eta\lambda + {*\chi} * \lambda\right)/g^2$$

- $U(1)_A$ current has an anomaly
  $$d^\dagger J_A = \frac{1}{4\pi}\mathscr{R}$$
  scalar curvature on Σ_h

  In particular, $\int d^\dagger J_A \omega = 2 - 2h = \chi_h$

  Euler characteristic of Σ_h

$U(1)_V$ current

- We call another global symmetry $U(1)_V$

  $$\delta_V \Psi = \theta_V \gamma_V \Psi$$

  where

  $$\gamma_V = \begin{pmatrix} 0 & 0 & -* \\ 0 & -* & 0 \\ \omega & 0 & 0 \end{pmatrix}$$

  $\eta \leftrightarrow *\chi, \lambda \leftrightarrow *\lambda, Q \leftrightarrow \tilde{Q}$, etc.

- Associated $U(1)_V$ current is given by

  $$J_V = (*\chi\lambda - \eta * \lambda)/g^2$$

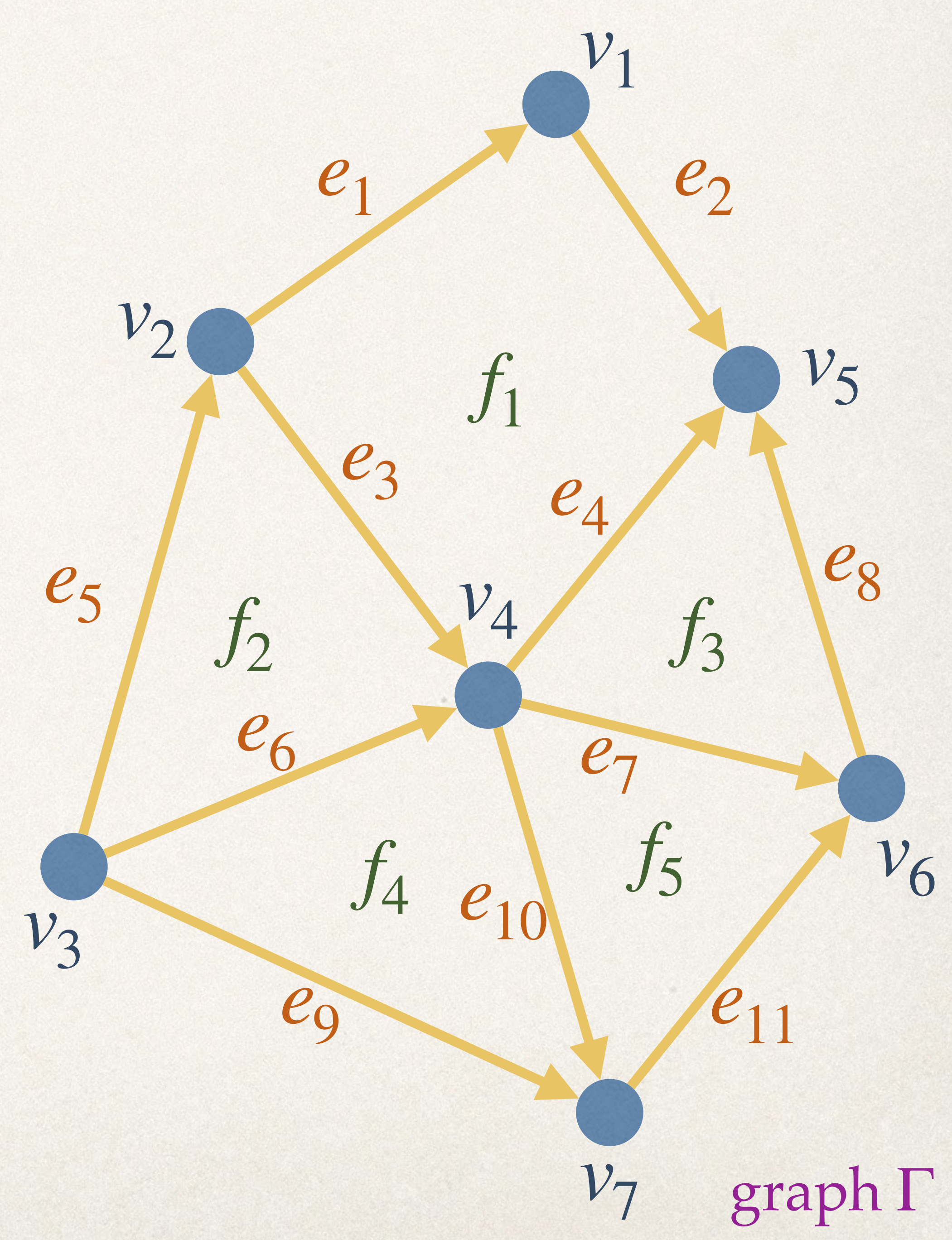
- $U(1)_V$ current associates with supercurrents J_Q and $J_{\tilde{Q}}$

  $$QJ_V = J_{\tilde{Q}}, \quad \tilde{Q}J_V = -J_Q$$

  So we find that $d^\dagger J_V = 0 \Rightarrow d^\dagger J_Q = d^\dagger J_{\tilde{Q}} = 0$

Graph

- A (connected and directed) graph Γ consists of vertices V and edges E
- We also consider faces F, which are surrounded by closed edges
- A dual graph Γ^* is defined by exchanging V and F (also E and E^*)



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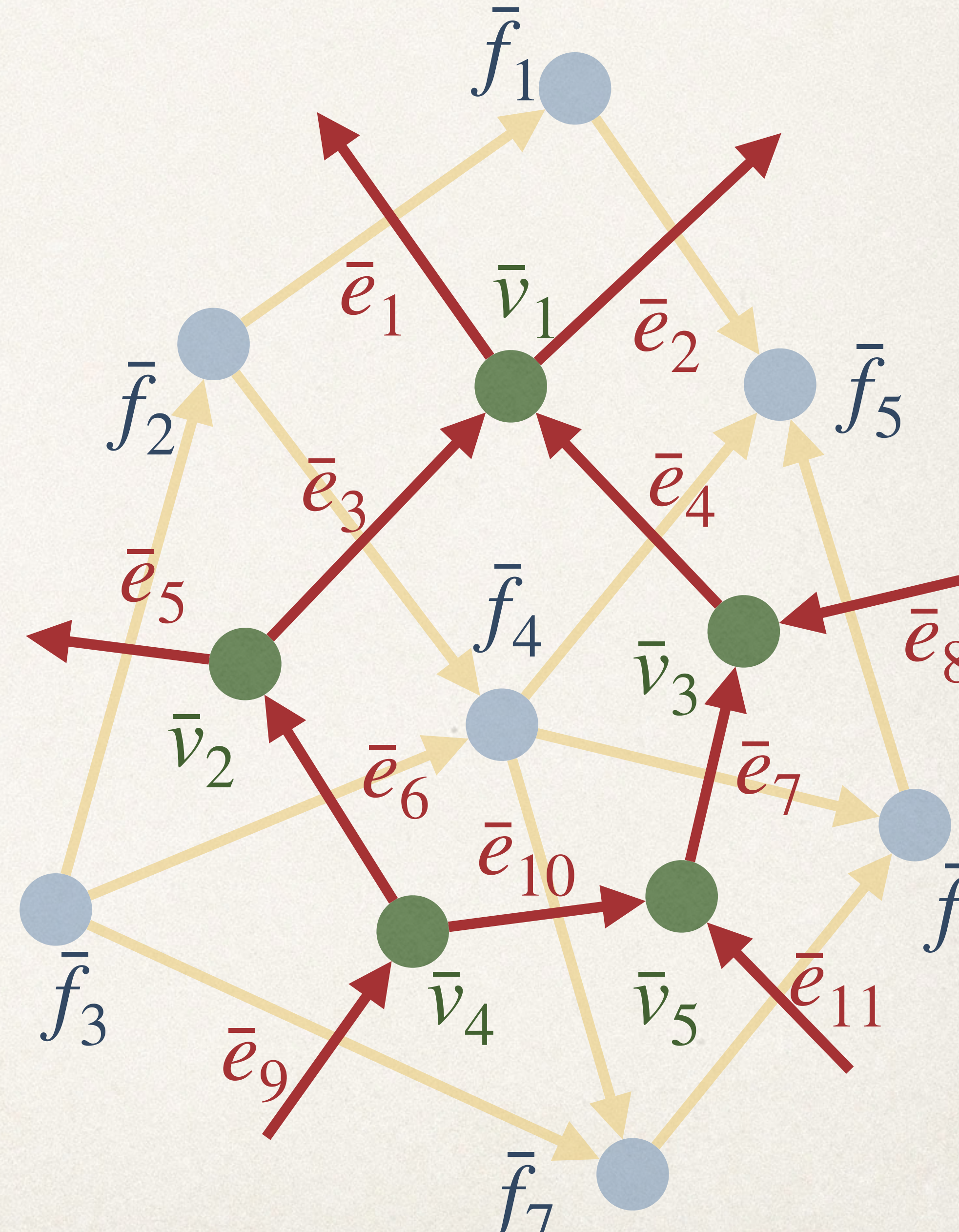


dual graph Γ^*

Differential forms and graph

- There is a good correspondence between the differential forms (fields) on the Riemann surface Σ_h and the objects on the graph Γ

	Differential forms	Fields	Graph objects	Variables
Bosons	0-form	$\phi,\ \bar{\phi}$	Vertex	$\phi^v,\ \bar{\phi}^v$
	1-form	A	Edge	$U^e \equiv e^{iA^e}$
	2-form	Y	Face	Y^f
Fermions	0-form	η	Vertex	η^v
	1-form	λ	Edge	$\Lambda^e \equiv e^{i\lambda^e}$
	2-form	χ	Face	χ^f

Differential forms and graph

- We can define the SUSY on the graph as well as the cont. field theory

$$
\begin{aligned}
&Q\phi = 0, && \\
&Q\bar{\phi} = 2\eta, && Q\eta = 0 \\
&QA = \lambda, && Q\lambda = -d\phi \\
&QY = 0, && Q\chi = Y
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
&Q\phi^v = 0, && \\
&Q\bar{\phi}^v = 2\eta^v, && Q\eta^v = 0 \\
&QA^e = i\lambda^e, && Q\lambda^e = -L^e{}_v\phi^v \\
&QY^f = 0, && Q\chi^f = Y^f
\end{aligned}
$$

where $L^e{}_v$ is an incidence matrix on the graph

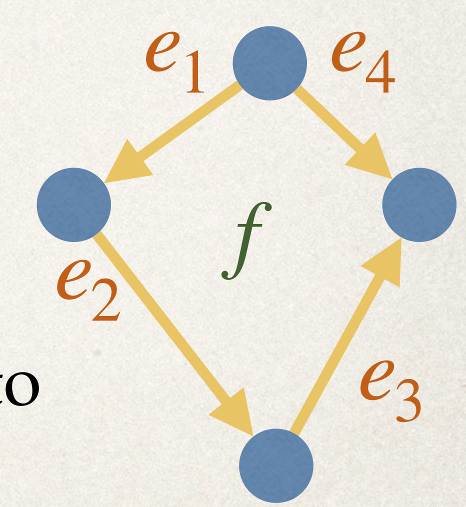
- The action also can be written in the Q-exact form:

$$S = -\frac{1}{2g_0^2} Q\left[\bar{\phi}_v (L^T)^v{}_e \lambda^e + \chi_f (Y^f - 2\Omega^f)\right]$$

where Ω^f is a function of the plaquette (face) variable, which goes to

$$\Omega^f \equiv -\frac{i}{2}\left(U^f - U^{f\dagger}\right) \rightarrow F$$

in the continuum limit

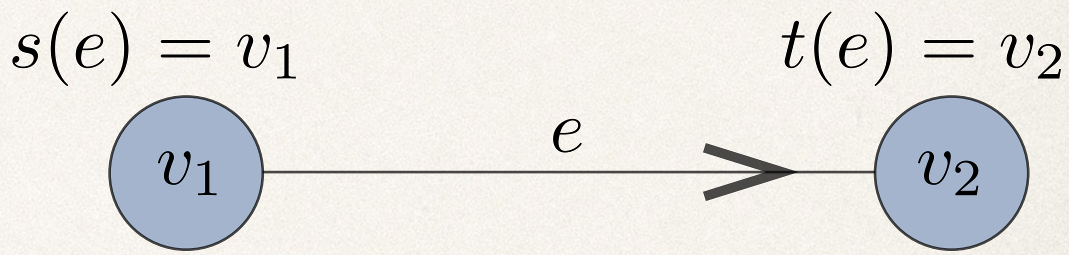


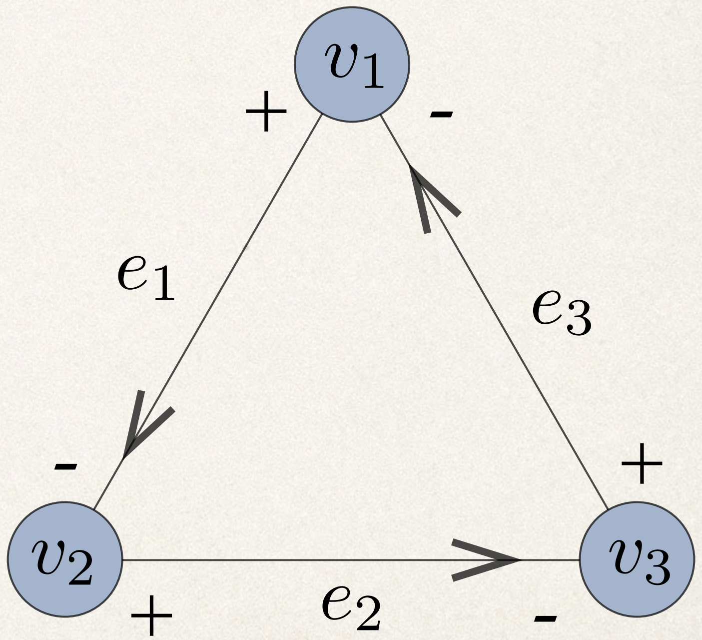
$$U^f = U_1 U_2 U_3 U_4^{-1}$$

Incidence matrix

- Incidence matrix L: $V(\Gamma) \to E(\Gamma)$ ($n_e \times n_v$ matrix)

$$L^e{}_v = \begin{cases} +1 & \text{if } s(e) = v \\ -1 & \text{if } t(e) = v \\ 0 & \text{others} \end{cases}$$





e.g.

$$L(\Gamma) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} +1 & -1 & 0 \\ 0 & +1 & -1 \\ -1 & 0 & +1 \end{pmatrix} \end{array}$$

Known as charge matrix (toric data) for the bi-fundamental matters in quiver gauge theory

SUSY action on the graph

- Bosonic part of the SUSY action:

$$S_b = \frac{1}{2g_0^2}\left[\bar{\phi}_v L^{Tv}{}_e L^e{}_{v'}\phi^{v'} - Y_f(Y^f - 2\Omega^f)\right]$$

$$\Rightarrow \frac{1}{2g_0^2}\left[\bar{\phi}_v(\Delta_V)^v{}_{v'}\phi^{v'} + \Omega_f\Omega^f\right] \text{ where } \Delta_V \equiv L^T L \text{ is the graph Laplacian}$$

- Fermionic part of the SUSY action:

$$S_f = \frac{1}{2g_0^2}\Psi^T i\not{D}\Psi$$

where

$$\Psi = \begin{pmatrix} \eta^v \\ \lambda^e \\ \chi^f \end{pmatrix}, \quad i\not{D} = \begin{pmatrix} 0 & -L^T & 0 \\ L & 0 & D \\ 0 & -D^T & 0 \end{pmatrix}, \quad (D^T)^f{}_e \equiv \frac{\delta\Omega^f}{\delta A^e} \propto (\check{L}^T)^f{}_e$$

incidence matrix on the dual graph Γ^*

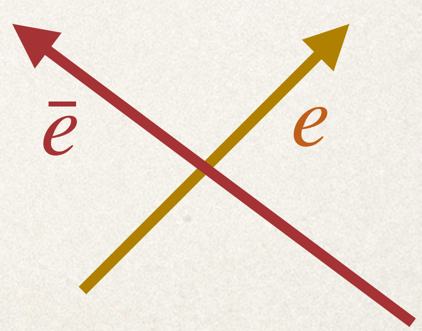
Properties of the "Dirac operator"

- We can see the correspondence between the (co)differentials and incidence matrices

$$i\not{D}(\Sigma_h) = \begin{pmatrix} 0 & -d^\dagger & 0 \\ d & 0 & d^\dagger \\ 0 & -d & 0 \end{pmatrix} \Rightarrow i\not{D}(\Gamma) = \begin{pmatrix} 0 & -L^T & 0 \\ L & 0 & D \\ 0 & -D^T & 0 \end{pmatrix}$$

- $\not{D}$ is a square root of the graph Laplacians

$$\not{D}^2 = \begin{pmatrix} L^TL & 0 & 0 \\ 0 & LL^T + DD^T & 0 \\ 0 & 0 & D^TD \end{pmatrix} \equiv \begin{pmatrix} \Delta_V & 0 & 0 \\ 0 & \Delta_E & 0 \\ 0 & 0 & \Delta_F \end{pmatrix}$$



where we have used the orthogonality between L and D:

$$L^TD = D^TL = 0 \quad \text{(corresponds to } d^2 = d^{\dagger 2} = 0\text{)}$$

SUSY on the dual graph

- We can dualize the SUSY on Γ to one on the dual graph Γ^*

$$
\begin{aligned}
&Q\phi^v = 0, \\
&Q\bar{\phi}^v = 2\eta^v, \quad Q\eta^v = 0 \\
&QA^e = \lambda^e, \quad Q\lambda^e = -L^e{}_v\phi^v \\
&QY^f = 0, \quad Q\chi^f = Y^f
\end{aligned}
\qquad
\begin{aligned}
&\tilde{Q}\phi^f = 0, \\
&\tilde{Q}\bar{\phi}^f = 2\chi^f, \quad \tilde{Q}\chi^f = 0 \\
&\tilde{Q}A^e = \lambda^e, \quad \tilde{Q}\lambda^e = -\check{L}^e{}_f\phi^f \\
&\tilde{Q}Y^v = 0, \quad \tilde{Q}\eta^v = Y^v
\end{aligned}
$$

where we have used the relation $\bar{v} = f, \bar{e} = e, \bar{f} = v$

- The dual action is defined as the $\tilde{Q}$-exact form:

$$\tilde{S} = -\frac{1}{2g_0^2}\tilde{Q}\left[\lambda_{\bar{e}}\check{L}^{\bar{e}}{}_f\bar{\phi}^f + \eta_v(Y^v - 2\Omega^v)\right]$$

where $\Omega^v \equiv M^v{}_f\Omega^f$, etc., but we find

$$S \neq \tilde{S}, \{Q, \tilde{Q}\} \neq 0, \tilde{Q}S \neq 0$$

Q, $\cancel{Q_I}$, $\cancel{\tilde{Q}}$

on the graph

$U(1)_V$ violation

- Unlike on the Riemann surface (Hodge duals), there is no symmetry under exchanging the vertices and faces
- There preserves only one supercharge Q and $U(1)_V$ is violated on the graph Γ
- We can show that the $U(1)_V$ symmetry does not have a quantum anomaly

  $\langle \partial J_V \rangle = \langle \Psi^T \gamma_V \, \not{D} \Psi \rangle$ where γ_V is traceless

- We expect that the $U(1)_V$ is restored in the continuum limit

$U(1)_A$ current and anomaly

- $U(1)_A$ current on the graph is given by

$$\partial J_A = \frac{1}{2g_0^2}\Psi^T i\gamma_A \not{D}\Psi$$

where

$$\gamma_A = \begin{pmatrix} \mathbf{1}_V & 0 & 0 \\ 0 & -\mathbf{1}_E & 0 \\ 0 & 0 & \mathbf{1}_F \end{pmatrix}$$

thus

$$\begin{aligned} \langle \partial J_A \rangle &= \langle \Psi^T i\gamma_A \not{D}\Psi \rangle / 2g_0^2 \\ &= \mathrm{Tr}_{V\oplus E\oplus F}\gamma_A \\ &= \dim V - \dim E + \dim F = \chi_\Gamma \end{aligned}$$

All processes are finite unlike continuous field theory

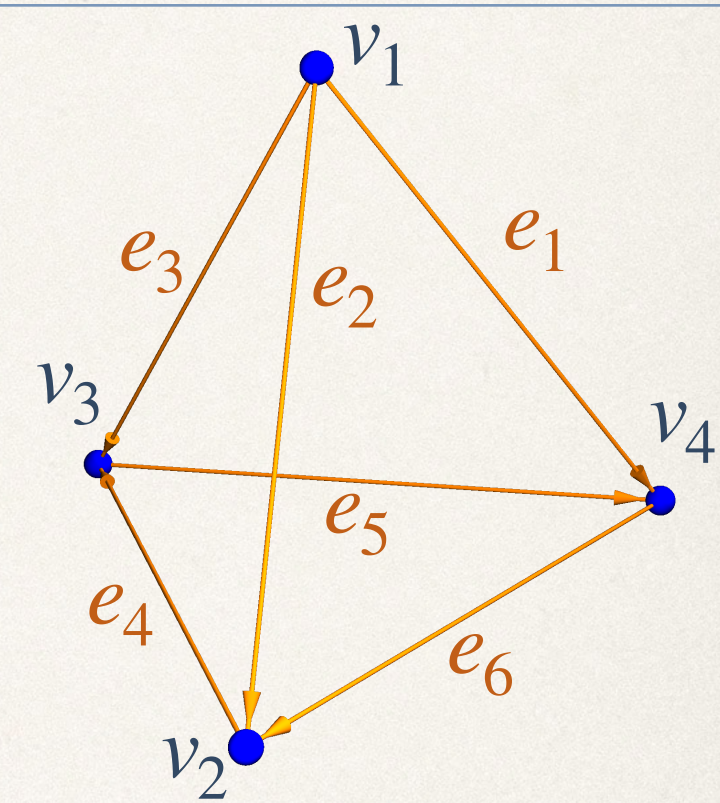
Heat kernel regularization

- Let us introduce the following heat kernel
$$h(t)^x{}_y \equiv e^{-t\not{D}^2} \quad \text{where } \ x, y \in V, E, F$$
which obeys the heat equation
$$\left(\frac{\partial}{\partial t} + \not{D}^2\right) h(t) = 0$$
- Using the eigenvectors of $\not{D}^2$, we obtain a trace of the heat kernel
$$\tilde{h}(t) \equiv \sum_n \Psi_n^T h(t) \Psi_n = \sum_n e^{-t\lambda_n^2}$$
eigenvalues
- We evaluate the $U(1)_A$ current by
$$\langle \partial J_A \rangle = \mathrm{Tr}_{V\oplus E\oplus F} \gamma_A e^{-t\not{D}^2} = \mathrm{Tr}_V e^{-t\Delta_V} - \mathrm{Tr}_E e^{-t\Delta_E} + \mathrm{Tr}_F e^{-t\Delta_F}$$
$$= \mathrm{ind}\ \not{D}$$

Examples: Tetrahedron

- Incidence matrix on the tetrahedron is given by

$$L = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \check{L} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$



- Laplacians are

$$\Delta_V = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}, \Delta_E = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \Delta_F = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Spec $\Delta_V = \{4,4,4,0\}$, Spec $\Delta_E = \{4,4,4,4,4,4\}$, Spec $\Delta_F = \{4,4,4,0\}$

Thus we find $\mathrm{Tr}_{V\oplus E\oplus F\oplus}\gamma_A e^{-t\mathbb{D}^2} = (3e^{-4t}+1) - 6e^{-4t} + (3e^{-4t}+1) = 2$

Index theorem on polyhedra

	Spec Δ_V	Spec Δ_E	Spec Δ_F	$\mathrm{Tr}_{V\oplus E\oplus F}\, \gamma_A e^{-t\mathbb{D}^2}$
Tetrahedron	{4,4,4,0}	{4,4,4,4,4,4}	{4,4,4,0}	$(3e^{-4t}+1)$ $-6e^{-4t}$ $+(3e^{-4t}+1)=2$
Hexahedron	{6,4,4,4,2,2,2, 0}	{6,6,6,4,4,4,4, 4,4,2,2,2}	{6,6,4,4,4,0}	$(e^{-6t}+3e^{-4t}+3e^{-2t}+1)$ $-(3e^{-6t}+6e^{-4t}+3e^{-2t})$ $+(2e^{-6t}+3e^{-4t}+1)=2$
3×3 torus	{6,6,6,6,3,3,3, 3,0}	{6,6,6,6,6,6,6, 6,3,3,3,3,3,3,3 ,3,0,0}	{6,6,6,6,3,3,3, 3,0}	$(4e^{-6t}+4e^{-3t}+1)$ $-(8e^{-6t}+8e^{-3t}+2)$ $+(4e^{-6t}+4e^{-3t}+1)=0$

Heat kernel on the graph

- Let us consider a subspace of the heat kernel

$$h_V(t)^{v}{}_{v'} \equiv e^{-t\Delta_V/a^2} \quad \text{where } v, v' \in V,\ a\text{: edge length}$$

- On the continuous 2d space-time, the heat kernel behaves

$$h(x,y;t) = \frac{1}{4\pi t} e^{-|x-y|^2/2t} + \cdots$$

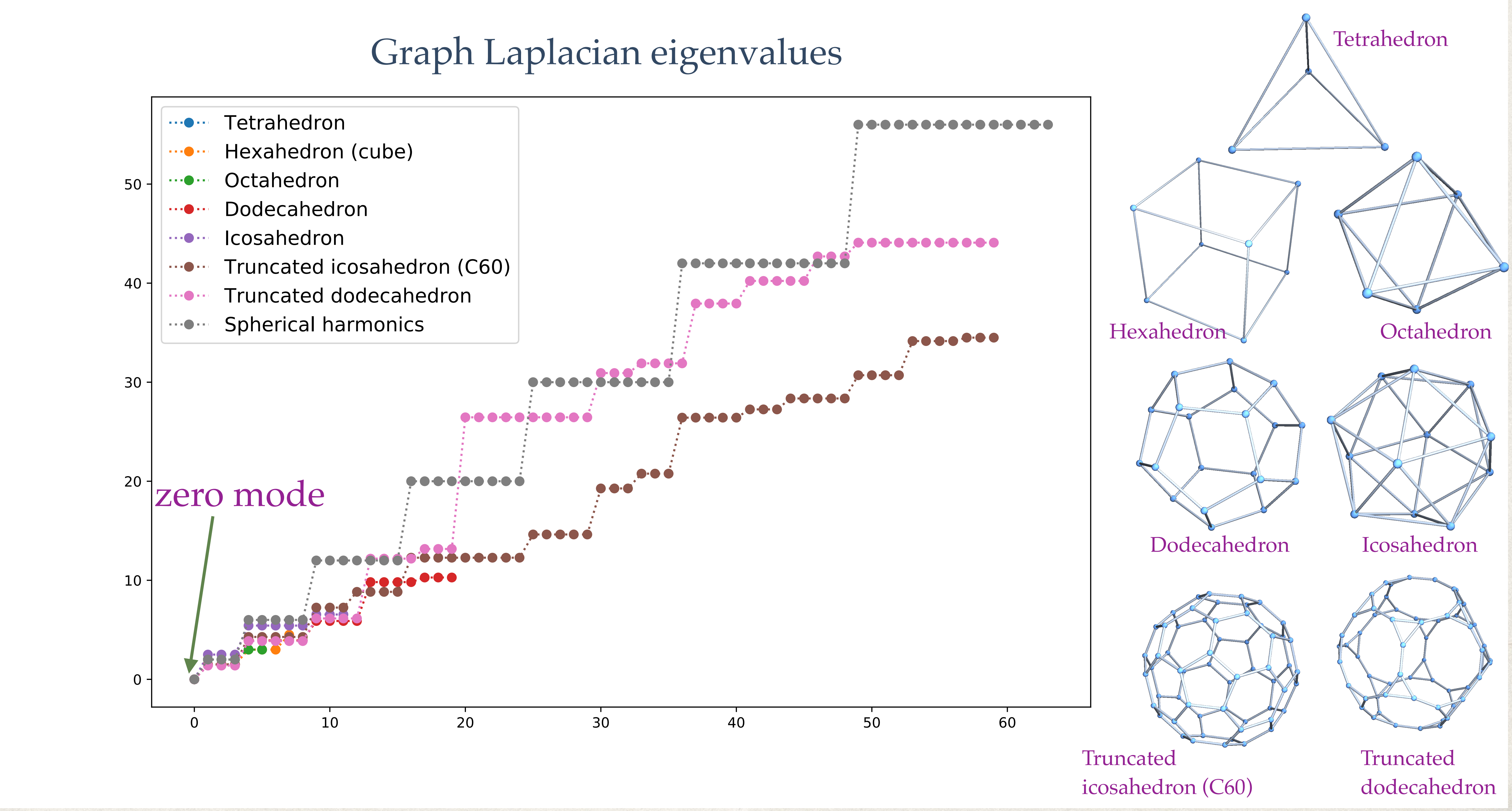
- On the other hand, the trace of the heat kernel gives

$$\tilde{h}(t) \equiv \int dx\, h(x,x;t) = \sum_n e^{-t\lambda_n^2}$$

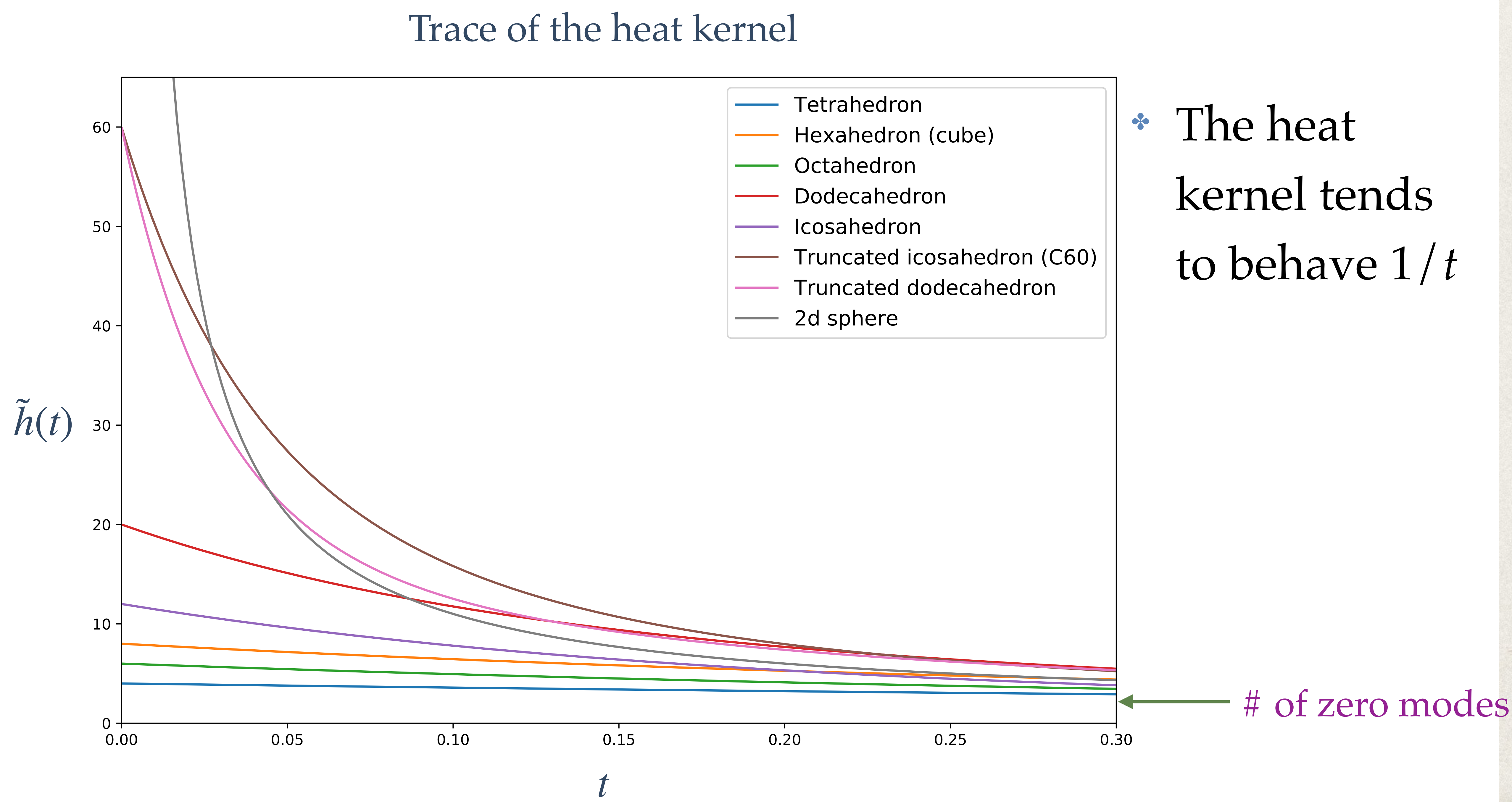
- We can compare the heat kernel on S^2 with the eigenvalues of the graph Laplacian with $\chi_\Gamma = 2$

$$\tilde{h}(t) = \frac{R^2}{t} + \cdots \quad \leftrightarrow \quad \tilde{h}_V(t) = \mathrm{Tr}_V e^{-t\Delta_V/a^2} \quad \text{where } R\text{: radius}$$

Comments on graph spectrum



Asymptotic behavior of the graph heat kernel



- The heat kernel tends to behave $1/t$

BRST symmetry

- We can introduce the ghosts, Nakanishi-Lautrup field and BRST transformation by

$$\delta_B c^v = 0, \quad \delta_B \bar{c}^v = 2B^v \quad \delta_B B^v = 0,$$
$$\delta_B A^e = -L^e{}_v c^v, \quad \delta_B \phi = 0, \text{ etc.}$$

- The ghost c is a superpartner of ϕ

$$Qc^v = \phi^v, \;\; Q\bar{c} = QB = 0$$

- We choose the gauge fixing function as

$$f^v = (L^T)^v{}_e A^e - \frac{1}{2}B^v \quad \text{(Coulomb gauge)}$$

$Q_B^2 = 0$
nilpotent

- If we define a combination of the SUSY and BRST symmetry by $Q_B \equiv Q - \delta_B$, the gauge fixing action is written in a Q_B-exact form

$$S' = -\frac{1}{2g_0^2} Q_B \left[\bar{\phi}^v (L^T)^v{}_e \lambda^e + \chi_f (Y^f - 2\Omega^f) + \bar{c}_v f^v \right] = S + S_{\text{GF+FP}}$$

Boson/Fermion correspondence

- Up to the 1-loop approximation, the gauge fixing action consists of

$$S'_b \sim \frac{1}{2g_0^2}\left[\bar{\phi}L^T L\phi + V^T X V\right]$$

$$S'_f = \frac{1}{2g_0^2}\left[\bar{c}L^T Lc + \Psi^T i\not{D}\Psi\right]$$

where

$$V = \begin{pmatrix} B^v \\ A^e \\ Y^f \end{pmatrix}, \quad X = \begin{pmatrix} -1 & L^T & 0 \\ L & 0 & D \\ 0 & D^T & -1 \end{pmatrix} \Longleftrightarrow \Psi = \begin{pmatrix} \eta^v \\ \lambda^e \\ \chi^f \end{pmatrix}, \quad i\not{D} = \begin{pmatrix} 0 & -L^T & 0 \\ L & 0 & D \\ 0 & -D^T & 0 \end{pmatrix}$$

- X and $\not{D}$ have the same determinant

  $\Rightarrow$ 1-loop determinants are canceled with each other except for zero modes

Conclusion and Discussion

Results:

- We found the correspondence between the differential forms and objects on the graph, and the (co)differential and (dual) incidence matrix on the graph
- The zero modes and anomaly are much similar to the continuous field theory

Outlook:

- Inclusion of the chiral superfields (a generalization of Hirzebruch–Riemann–Roch theorem, chiral anomaly)
- Extension to higher dimensional manifold
- Check by the numerical simulation