幾何学の意味での量子化の一般化

佐古 彰史 (理科大)

郷原 惇平 (理科大), 廣田 祐士 (麻布大)

離散的手法による場と時空のダイナミクス

2019年9月9日 島根大学

arXiv 190708665, 190902361

1. Intro. & Motivation

- Classical Mechanics $\xrightarrow{\text{Quantization}}$ Quantum Mechanics

  $\Uparrow$

  several ways of Quantization
  - canonical quantization (prequantization)
  - path integral
- Classical Field theory $\longrightarrow$ Quantum field theory
- Classical Gravity $\longrightarrow$? String theory

  ? loop-gravity

  ? Matrix model

How can we generalize and restrict the way of quantization ?

FACT.

Fiber Bundle

$$
\begin{array}{l}
E \ : \text{Vector bundle} \\
\downarrow \pi \\
M : \text{Riemanian mfd} \\
\qquad \| \\
\qquad \text{Classical Gravity}
\end{array}
$$

with connections

+ Q.M

(Q.E.D)

(Q.C.D).

$\longrightarrow$ Q. Gauge theory

$\uparrow\downarrow$

$\longrightarrow$ String?

$\uparrow$

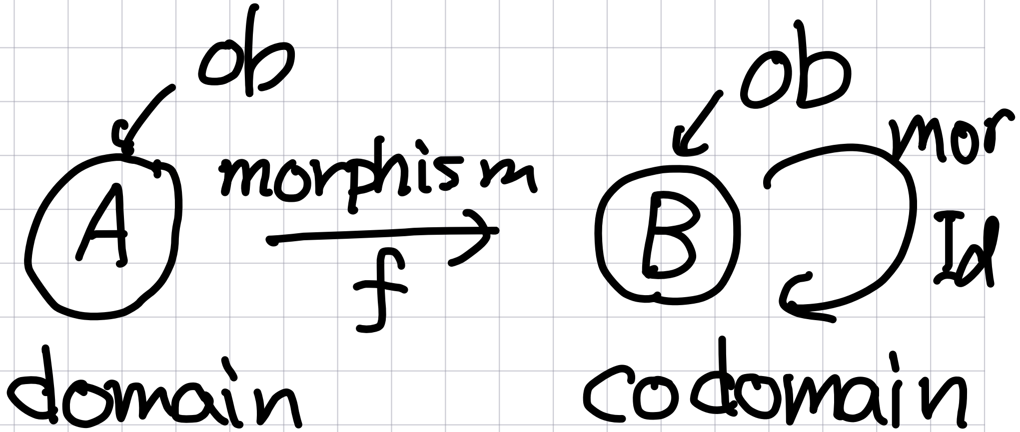
$\longrightarrow$ Matrix model

$\longrightarrow$?

To describe a such map,
"category" is the most natural tool.

~ Category ~

- morphism, object



Def). Category $\mathcal{C}$

$ob(\mathcal{C})$: set of objects.

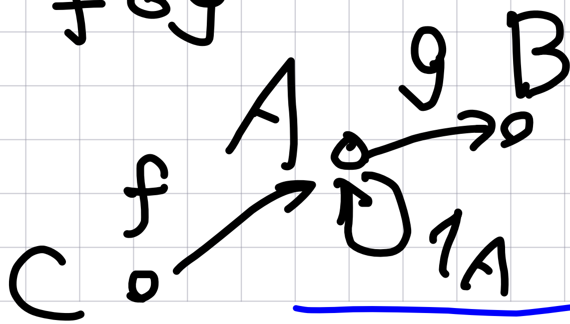
$Mor(\mathcal{C})$: set of morphisms.

Set of mor $\mathcal{C}(A,B)$: $A \to B$

- $f \circ g \in Mor(\mathcal{C})$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

- $(f \circ g) \circ h = f \circ (g \circ h)$
- $\forall A \in ob(\mathcal{C})\ \exists 1_A$
  s.t. $g = g \circ 1_A$, $f = 1_A \circ f$



~Functor, Natural transformation~

Def. Functor $F \quad \mathcal{C} \longrightarrow \mathcal{D} \quad (\mathcal{C}, \mathcal{D} : \text{category})$

$\left(ob(\mathcal{C}) \to ob(\mathcal{D}),\quad Mor(\mathcal{C}) \longrightarrow Mor(\mathcal{D}) \right)$

$\mathcal{C}$ $\qquad\qquad\qquad\qquad$ $\mathcal{D}$

① $\quad A \xrightarrow{f} B \qquad \xrightarrow{F} \qquad F(A) \xrightarrow{F(f)} F(B)$

② $\quad A \xrightarrow{f} B \xrightarrow{g} C \qquad \xrightarrow{F} \qquad F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$

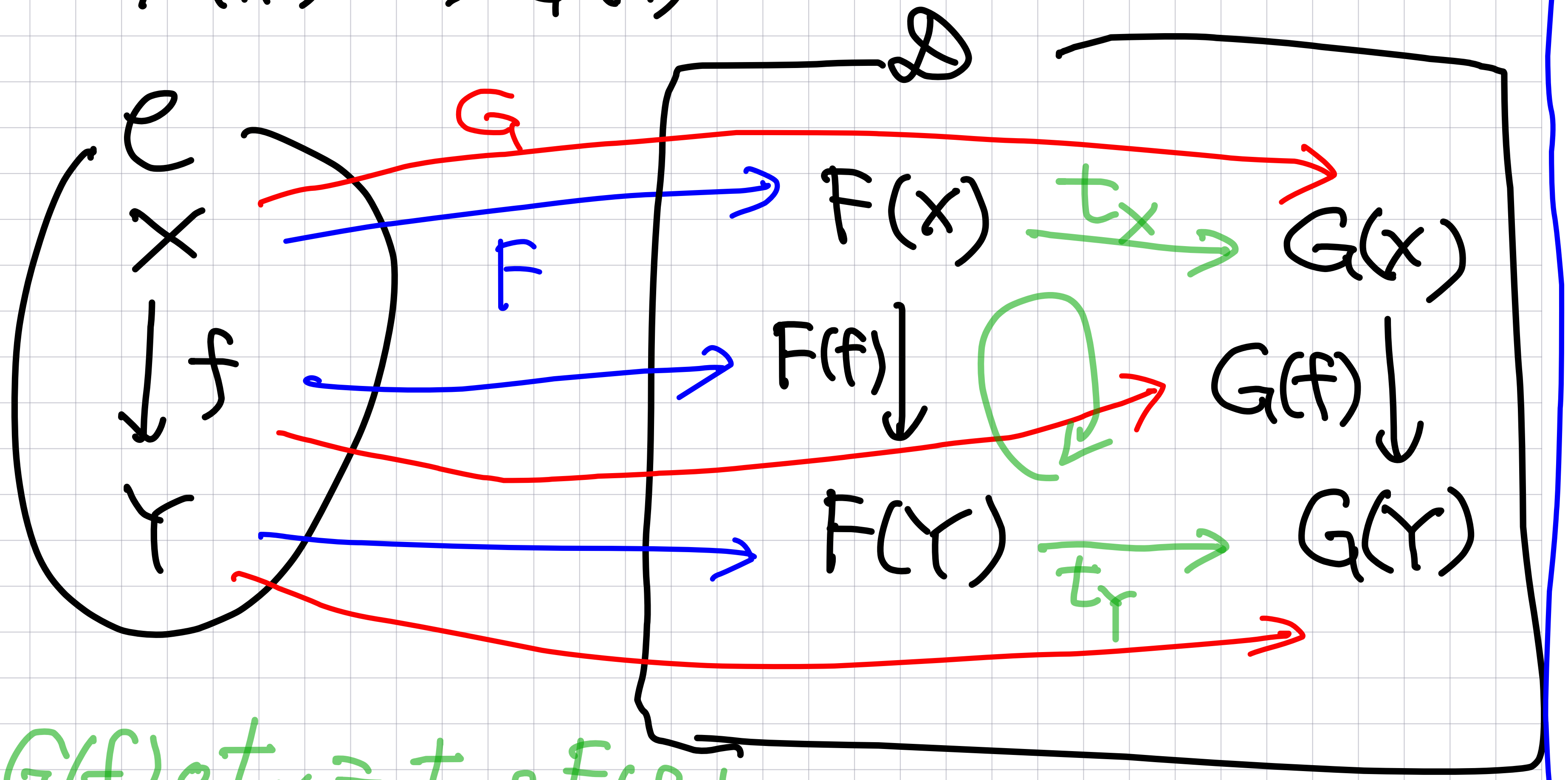
③ $\quad A \circlearrowleft 1_A \qquad \xrightarrow{F} \qquad F(A) \circlearrowleft 1_{F(A)} = F(1_A)$

Def. Natural transformation $F \overset{t}{\Longrightarrow} G$

(F, G : functor $\mathcal{C} \to \mathcal{D}$)

t : Set of $\mathrm{Mor}(\mathcal{D})$

$F(X) \to G(X)$



$$G(f) \circ t_X = t_Y \circ F(f)$$

~ Quantizations ~

• Dirac.

$$\hat{\ } : f \in C^\infty(M) \longrightarrow \hat{f} \in \mathrm{End}(\mathcal{H})$$

$$(1)\ \hat{H}_1 + \hat{H}_2 = \widehat{H_1 + H_2}, \quad (2)\ \widehat{\lambda H} = \lambda \hat{H}$$

$$(3)\ [\hat{H}_1, \hat{H}_2] = i\,\widehat{\{H_1, H_2\}}, \quad (4)\ \hat{1} = \mathrm{Id}$$

There is no Perfect Quantization

• Deformation Quantization

De Wilde-Lecomte, Fedsov, Omori-Maeda-Yoshioka, Kontsevich, etc

• Matrix regularization

Belezin, Toeplitz, Hoppe, de Wit, Nicolai. etc.

$$\cancel{(3)\ [\hat{H}_1, \hat{H}_2] = \widehat{\{H_1, H_2\}}}$$

• Geometric Qun.

Weyl, Kostant, Souriau, etc

~~Observable~~

It is convinient if there is a perspective unifying these quantizations!

2. Def. of Quantization Category

▷ Preparation.

"limit" is used as the following meaning.

$\mathcal{C}$: category

J : index category

$$① \xrightarrow{f_{12}} ② \xrightarrow{f_{23}} ③$$

$$J \xrightarrow{F} \mathcal{C}$$

diagram

$$\begin{tikzcd}
 & N \arrow[d, dashed, "\text{Unique}"] \arrow[ddl] \arrow[dd, bend left] \arrow[ddr] & \\
 & \boxed{L}\ \text{Limit} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\
\boxed{1} = F① \arrow[r, "F(f_{12})"'] & \boxed{2} = F② \arrow[r, "F(f_{23})"'] & \boxed{3} = F③
\end{tikzcd}$$

(L, π) is called "limit"

Def). Pre-2 category. $\mathcal{C}$

RMod : category of R-module over com. R.

$A(M)$: Poisson alg. $(C^\infty(M), \cdot, \{,\})$

fixed

M is a Poisson mfd.

Not alg.

Mor($\mathcal{C}$) is linea fun.

$\mathcal{C}$: sub Category of RMod s.t.

1. $A(M) \in ob(\mathcal{C})$

2. $^\forall M_i \in ob(\mathcal{C})$ is a Lie alg $([\ ,\ ]_i)$.

3. $^\exists T_i \in \mathcal{C}(A(M), M_i)$ s.t.

$$[T_k(f), T_k(g)]_k = i\hbar(T_k)\, T_k(\{f,g\}) + O(\hbar^{1+\epsilon}(T_k))$$

Def). Character χ

$$\chi : ob(\mathcal{C}) \to \mathbb{R}.$$

$$\chi(M_i) = \max_{T_i \in ob(A(M), M_i)} h(T_i)$$

m is # of connected components

Def). $J^\bullet$: index category $\quad J^\bullet = \coprod_{\alpha}^{m} J^\alpha \quad J^\alpha$: connected category

J^α

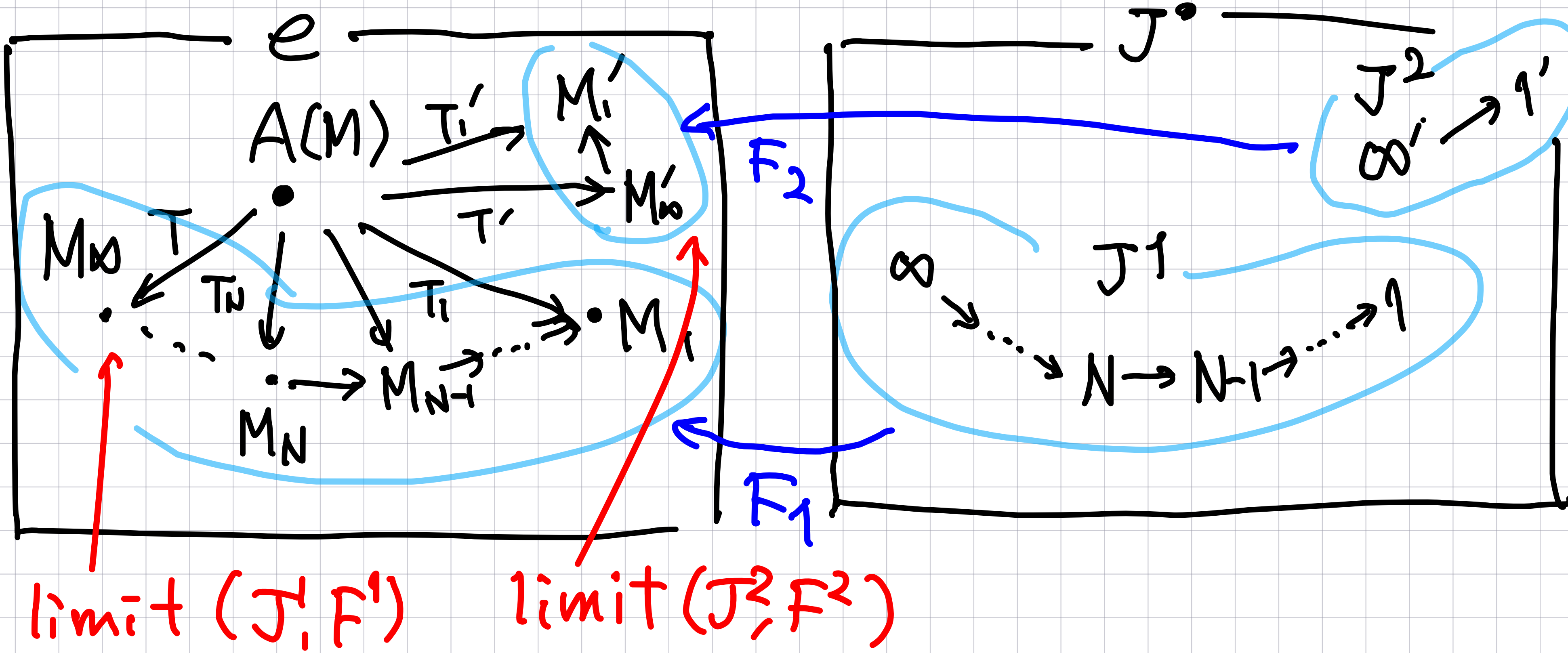
- $\forall M_i \in ob(\mathcal{C}) \setminus \{A(M)\}$ $\exists ! J^\alpha$, $i \in ob(J^\alpha)$
- $T_{ij}^k \in \mathcal{C}(M_i, M_j)$, $\chi(M_i) \leq \chi(M_j) \iff (i,j)^k \in J^\alpha(i,j)$

$F^\bullet$: set of diagonal functor $\quad F^\bullet = \{F^1, \cdots, F^m\}$

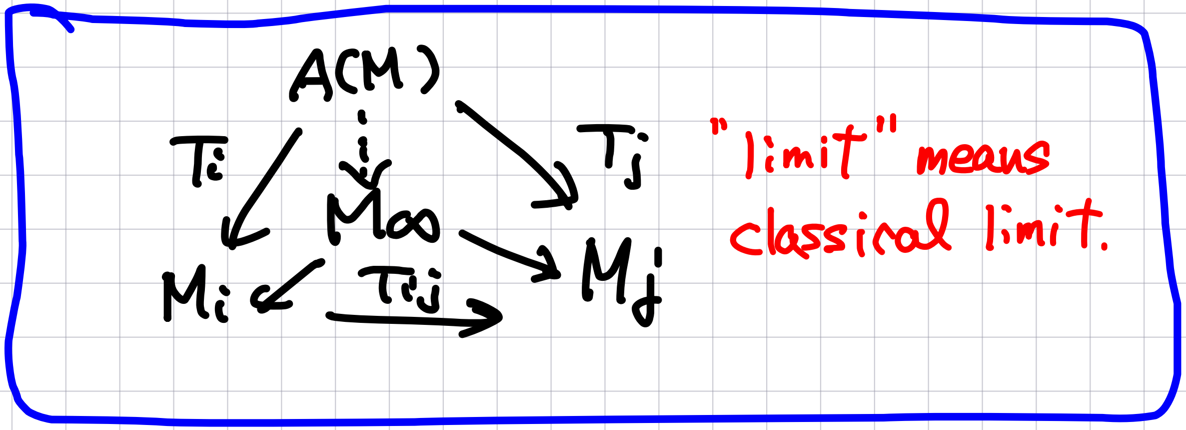
- $F^\alpha : J^\alpha \to \mathcal{C}$

object $i \mapsto F(i) = M_i$, $\quad$ Morphism $(i,j)^k \mapsto T_{ij}^k$

Prop. $\mathcal{C}$ has limit M_∞^α for each (J^α, F^α)



Note
$A(M)$ is a candidate for all (J^α, F^α), always.



Def). Quantization Category $\mathcal{Q}$ of Poisson alg $A(M)$.

$\mathcal{Q}(\mathcal{C}(M), J^0, F^0, \chi)$ is a category $\mathcal{C}(M)$ satisfying following conditions

- $\forall f, g \in A(M)$. $T := T_\infty \in \mathcal{C}(A(M), M_\infty)$ satisfies the following quantization conditions:

Q1. $T(fg) - T(f)T(g) = 0$

Q2. $[T(f), T(g)]_\infty - i\hbar\, T(\{f, g\}) = 0$

Q1 ~ Q2. are similar conditions with them in Berezine-Toeplitz quantization or Matrix regularization.

3. Matrix Regularization (including B-T quantization)

Def). $\mathcal{C}_{MR}$: pre-2 category for Matrix regularization

$\{N_i\}$: strictly increasing sequence of $\mathbb{N}$

$\hbar$: strictly decreasing fun. s.t. $\lim_{N\to\infty} N\hbar(N)$ converges

$\mathcal{C}_{MR}(M)$ is defined as follows.

$N_k \times N_k$ Matrix alg.

- $ob(\mathcal{C}_{MR}(M)) = \{A(M), Mat_{N_k} (k=1,2,\cdots), Mat_\infty\}$
- $Mor(\mathcal{C}_{MR}(M))$: set of linear fun.

s.t. $\exists!\, T_i : A(M) \longrightarrow Mat_{N_i}$, $\exists!\, T_{ij} : Mat_{N_i} \longrightarrow Mat_{N_j}$

with $T_i = T_{ij} \circ T_j$

$$J^{\circ}_{MR}: \quad \begin{tikzcd} 1 & 2 \arrow[l, "f_{12}"'] & 3 \arrow[l, "f_{2}"'] & \cdots \end{tikzcd}$$

$j \to i$ for $i \le j$

$$\xrightarrow{F}$$

$$\mathcal{C}_{MR} \quad \begin{tikzcd} & & A(M) \arrow[dll, "T_1"'] \arrow[dl, "T_2"] \arrow[d, dashed] \arrow[dr, "T_3"] & \cdots \\ & & M_\infty \arrow[dll, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[dr, "\pi_3"] & \\ M_{N_1} & M_{N_2} \arrow[l, "T_{12}"] & & M_{N_3} \arrow[ll, "T_{23}"] & \cdots \arrow[l] \end{tikzcd}$$

$\mathcal{X}_{MR}(M_{N_i}) = \hbar(N_i)$ determine J°_{MR}, F.

$$\boxed{\underline{\text{Thm.}}\ (\mathcal{C}_{MR}, J^{\circ}, F, \mathcal{X}) \text{ is } 2 \text{ category } (\mathcal{Q}_{MR}) \text{ with the limit } (M_\infty, \pi)}$$

4. Deformation Quantization

Let's review. D.Q.

Def). Deformation Quantization $(\mathcal{F}, *)$.

$\mathcal{F} := \{ f \mid f = \sum \hbar^k f_k ,\ f_k \in C^\infty(M) \}$ formal P.S.

$f * g = \sum_k \hbar^k C_k(f, g)$

1. $*$ is associative
2. C_k is bidifferential op.
3. $C_0(f, g) = fg$, $C_1(f, g) = \frac{1}{2} i \{f, g\}$ → $[f, g] = i\hbar \{f, g\} + O(\hbar^2)$
4. $f * 1 = 1 * f = f$

For arbitrary Poisson mfd M.
there exist $(\mathcal{F}, *)$.

Let's consider "Not" formal. ⇒ "Strict D.Q."
(Rieffel, etc)

Def). pre 2 category $\mathcal{C}_{DQ}$

$ob(\mathcal{C}_{DQ}) = \{A(M), (\mathcal{F}, *)\}$

Lie alg by $[f,g]_* = f*g - g*f$

$A(M) \xrightarrow[T = \pi]{} \mathcal{F}$

$\mathcal{C}_{DQ}$

$$J_{DQ}^{\circ} \xrightarrow{F_{DQ}} \begin{array}{c} A(M) \\ \downarrow \pi \\ \mathcal{F} \end{array}, \quad \chi_{DQ}(A(M)) = \hbar, \quad \chi_{DQ}(\mathcal{F}) = 0$$

Thm $(\mathcal{C}_{DQ}, J_{DQ}^{\circ}, F_{DQ}, \chi_{DQ})$ is 2-category with limit $(A(M), \pi)$

~ Equivalence of Categories ~

Def). Naturally Isomorphic

functor $F, G : \mathcal{C} \to \mathcal{D}$

Natural trans. $F \overset{t}{\Rightarrow} G$

$X \mapsto F(X)$
$G(X)$

$t_X : F(X) \to G(X)$

For $^\forall X$, t_X is isomorphic ($^\exists t_X^{-1}$ s.t. $t_X \circ t_X^{-1} = 1$, $t_X^{-1} \circ t_X = 1$)

$\overset{def}{\Longleftrightarrow}$ t is naturally iso.

Def) Equivalence of Categories

$\underset{\text{Equiv}}{\mathcal{C} \simeq \mathcal{D}} \overset{def}{\Longleftrightarrow} {}^\exists F : \mathcal{C} \to \mathcal{D}, {}^\exists G : \mathcal{D} \to \mathcal{C}$ s.t.

${}^\exists \varepsilon : FG \overset{\varepsilon}{\Rightarrow} I_{\mathcal{D}}, {}^\exists \eta : GF \overset{\eta}{\Rightarrow} I_{\mathcal{C}}$

natural iso

Identity functor

Similar diagram up to iso.

Thm. $\partial_{MR} \underset{\text{categorically equiv}}{\simeq} \partial_{DQ}$ for $A(M) \simeq M_\infty$

Ex). Bordemann, Meinrenken, Schlichenmaier, B.T.g.u cpt. Kähler mfd

Proof) $\partial_{MR}|_{Mat_{N_i}}$ $\qquad$ ∂_{DR}

$$
\begin{tikzcd}
 & A(M) \arrow[dl, "T_\infty"'] \arrow[d, "T_i"] \arrow[rr, shift left, "F"] & & A(M) \arrow[ll, shift left] \arrow[d, "T"] \arrow[loop above, "Id"] \\
M_\infty \arrow[r, "\pi_i"'] & Mat_{N_i} \arrow[rr, shift left, "F"] & & \mathcal{F} \arrow[ll, shift left, "G"]
\end{tikzcd}
$$

F, G

Natural transformation

$\theta : FG \to id_{\partial_{DQ}}$

$\theta' : GF \to id_{\partial_{MR}}$

$$
\begin{tikzcd}
GF(A(M)) \arrow[r, "\theta'_{A(M)}=Id"] \arrow[d, "GF(T)"', "=id"] & A(M) \arrow[d, "T_\infty"] \\
GF(M_\infty) \arrow[r, "\theta'_{M_\infty}=T_\infty"'] & M_\infty
\end{tikzcd}
$$

$GF(M_\infty) = A(M)$

$$
\begin{tikzcd}
GF(M_\infty) \arrow[r, "\theta'_{M_\infty}=T_\infty"] \arrow[d, "GF(\pi_i)"', "=T_i"] & M_\infty \arrow[d, "\pi_i"] \\
GF(Mat_{N_i}) \arrow[r, "\theta'_{Mat_{N_i}}=Id"'] & Mat_{N_i}
\end{tikzcd}
$$

$GF(M_\infty) = A(M)$, $GF(Mat_{N_i}) \simeq Mat_{N_i}$

//

5 Other Quantization

Def. PreQuantization

Set of Op acting on $\mathcal{H}$

$\mathcal{C}_{PQ}$: $ob(\mathcal{C}_{PQ}) = \{A(M),\ Q(A(M))\}$

$J_{PQ} \xrightarrow{F_{PQ}} Q(A(M))$

$\chi_{PQ}(A(M)) = \hbar, \quad \chi_{PQ}(Q(A(M))) = 0$

Thm. $(\mathcal{C}_{PQ}, J_{PQ}^{Q}, F_{PQ}, \chi_{PQ})$ is $\mathcal{Q}$ category. $\mathcal{Q}_{PQ}$ with limit $(A(M), Q)$

$\mathcal{Q}_{PQ}$

$$\begin{tikzcd}
 & A(M) \arrow[d, "Q"] \\
J_{PQ} \arrow[r, "F_{PQ}"] & Q(A(M))
\end{tikzcd}$$

Thm.

$$\mathcal{Q}_{PQ} \overset{\text{equiv.}}{\simeq} \mathcal{Q}_{DQ} \cong \mathcal{Q}_{MR}\big|_{A(M)\simeq M_\infty}$$

Thm. Poisson Enveloping Alg.

$(\mathcal{C}_{P.E}, J_{P.E}, F_{P.E}, \chi_{P.E})$ is $\mathcal{Q}$ with limit $\underline{(A(M), \pi)}$

$\mathcal{C}_{P.E}$

$(A(M), \cdot, \{,\})$ — Limit

$$\begin{tikzcd}
 & (A(M), \cdot, \{,\}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, shift left, "\pi"] \arrow[dd, shift right] & \\
(A(M), \cdot) \arrow[dr, "\alpha"] \arrow[ddr, "\alpha_1"] \arrow[dddr, "\alpha_2"'] & & (A(M), \{,\}) \arrow[dl, "\beta"'] \arrow[ddl, "\beta_1"'] \arrow[dddl, "\beta_2"] \\
 & U \arrow[d, dashed] & \\
 & U_1 \arrow[d, dashed] & \\
 & U_2 \arrow[d, phantom, "\vdots"] & \\
 & {} &
\end{tikzcd}$$

U (Poisson Enveloping Alg.

$F_{P.E}$

Colimit : Quantization

$J_{P.E}^{\bullet}$ $\quad$ $\chi_{P.E}$

$$\begin{tikzcd}
① \arrow[dr] \arrow[ddr] \arrow[dddr] & & ② \arrow[dl] \arrow[ddl] \arrow[dddl] & 0 \\
 & ③ \arrow[d] \arrow[dd, bend left] & & 1 \\
 & ④ & & 1 \\
 & ⑤ \arrow[d, phantom, "\vdots"] & & 1 \\
 & {} & &
\end{tikzcd}$$

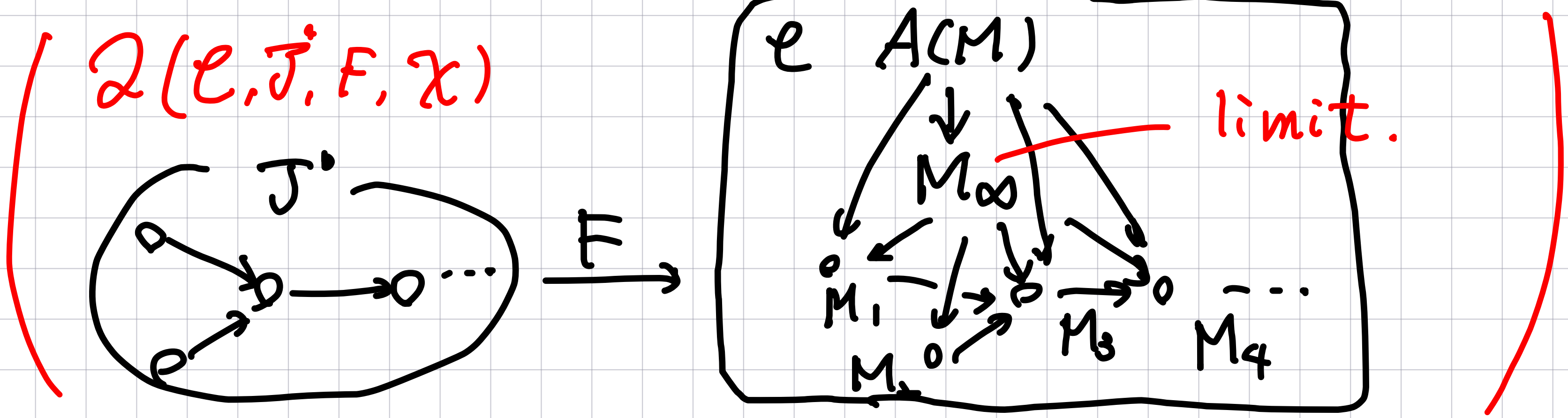
α, β is Ring hom

with $\alpha(\{a,b\}) = \beta(a)\alpha(b) - \alpha(b)\beta(a)$

$\beta(a,b) = \alpha(a)\beta(b) + \alpha(b)\beta(a)$

6. Conclusions.

We define $\mathcal{Q}$ as a generalization of Quantization.



Thm. Matrix reg, Def. Q, PreQ., Poissn E.A. are $\mathcal{Q}$

Thm. $\mathcal{Q}_{MR} \simeq \mathcal{Q}_{DQ} \simeq \mathcal{Q}_{P.Q}$. (for $A(M) = M_\infty$)

Poisson Enveloping $\mathcal{Q}$ is not equiv. with the others

$$\begin{array}{ccc} \mathcal{Q}_{P.E} \not\simeq & & \mathcal{Q}_{M.R.}|_{A(M)=M_\infty} \\ & \swarrow\!\!\wr & \wr\!\!\searrow \\ & \mathcal{Q}_{DQ} \simeq \mathcal{Q}_{P.Q}. & \end{array}$$

⇐ This is not shared with you today.

- Every theory of phys might be an object in a category like the Quantization category.

$\Downarrow$

- How does the nature chose one category as our universe ?

$\Downarrow$

Categorical extension of Hamiltonian eigenvalue problem ?

To be continued ...
Next Gohara's talk

Thank you for your attentions.