A review on M(atrix) theory, its covariantization and related problems

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(see also lecture note, arXiv: 1612.08513
and many previous works referred to, afterward)

- Matrix-theory approach to M theory
- Toward covariantization of ***DLCQ*** Matrix Theory
- Higher gauge symmetry from Nambu bracket
- 11d covariant action of Matrix theory
- Concluding discussions

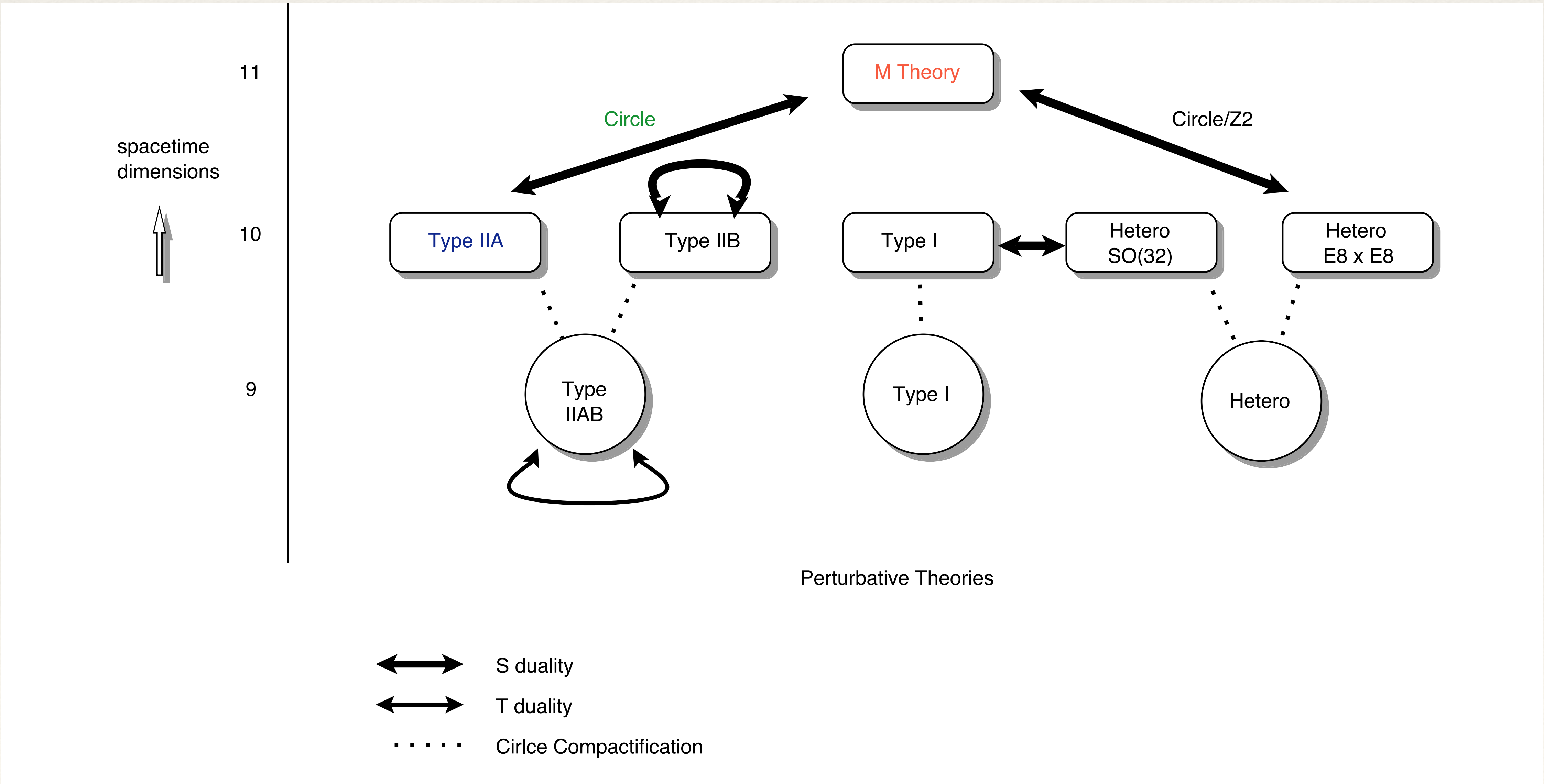
Prehistory :
Matrix-theory approach to M-theory

Basic tenets of M-theory conjecture   Hull-Townsend, Witten, ... 1995

- achieve a complete unification of strings and D-branes in a compactified 11 dimensional space-time
- fundamental length scales:

$$\ell_{11} = g_s^{1/3}\ell_s, \quad \ell_s = \sqrt{\alpha'} \quad (\alpha')^{-1} = \text{string tension}$$

$$R_{11} = g_s \ell_s : \text{compactification radius along the (spatial) 11th direction}$$



underlying physical degrees of freedom:

M2 branes ($\overset{dual}{\Leftrightarrow}$ M5-branes)

super-membranes

- wrapped along the 11th (circle) direction

$\longrightarrow$ (fundamental) strings in 10 dimensions

- extended within un-compactified 10 dimensions

$\longrightarrow$ D2-branes

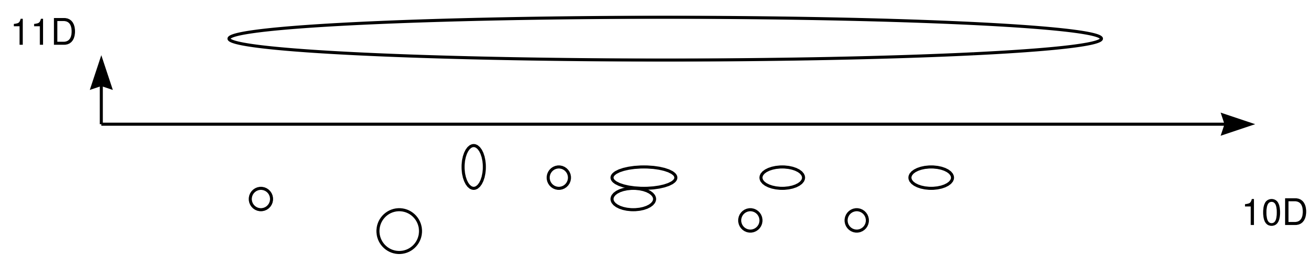
11D gravitons=10D gravitons + KK modes

strings D0-branes (particles)

The limit of un-compactification

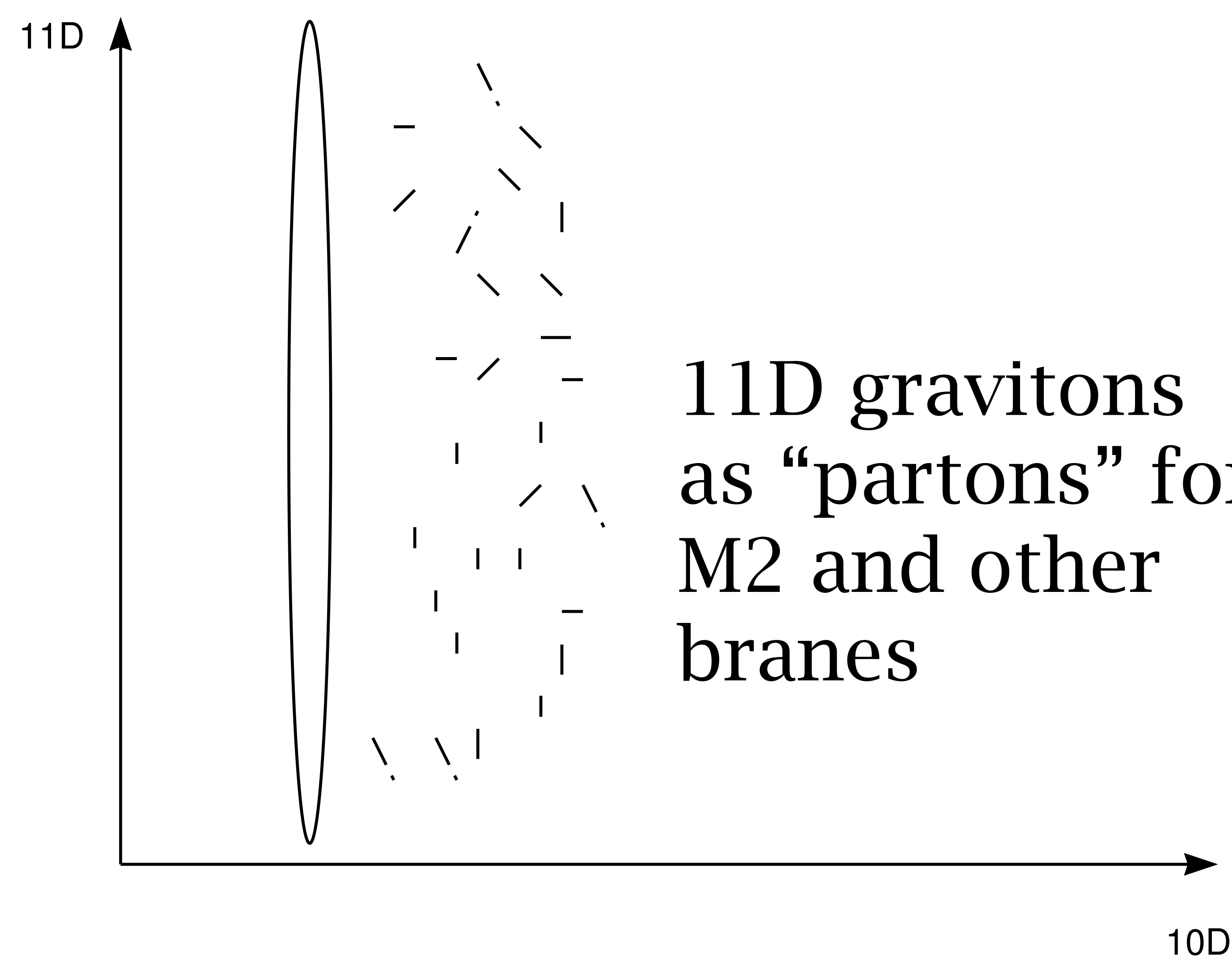
$$R_{11} = \ell_{11}^3/\ell_s^2 = \ell_{11} g_s^{2/3} \to \infty \text{ with fixed } \ell_{11} \Leftrightarrow \ell_s \to 0, \quad g_s \to \infty$$

corresponds to string theory with ***infinite string coupling*** and ***infinite string tension*** !



string-gravitons and D0 particles in 10D

limit of strong tension and coupling



11D gravitons as “partons” for M2 and other branes

Only tractable quantum formulation of (super) membranes

regularization in the light-cone gauge $X^+ = \xi^0 = \tau$

$$H = \frac{1}{2P^+}(P_a^2 + \frac{1}{2}\{X^a, X^b\}^2) + \cdots \qquad (a, b, \ldots) \in (1, 2, \ldots, 9)$$

$$\{X^a, X^b\} = \partial_1 X^a \partial_2 X^b - \partial_1 X^b \partial_2 X^b$$

U(N) gauge symmetric (super) SO(9) quantum mechanics

Goldstone-Hoppe,1982, ...

$$H = \frac{1}{2P^+}\mathrm{Tr}(\boldsymbol{P}_a^2 - \frac{1}{2}[\boldsymbol{X}^a, \boldsymbol{X}^b]^2) + \cdots \qquad \boldsymbol{P}^a = \frac{d\boldsymbol{X}^a}{dt} + i[\boldsymbol{A}, \boldsymbol{X}^a]$$

Combining with the fact that this coincides with the non-relativistic effective theory of D0-particles in 10D (type IIA) string theory, it is tempting to interpret this theory as the **light-cone gauge description of M-theory**.

M(atrix) theory conjecture Banks, Fischler, Shenker, Susskind, 1996

- Simple *but* quite non-trivial, well-defined and ***manifestly unitary quantum mechanics*** ;

However, ***non-covariant*** in the sense of 11d Lorentz symmetry

- Plenty of evidence, though not yet completely conclusive, with respect to correspondence with 11d supergravity;

mostly based on perturbative studies (*gravity=loop effect*);
graviton (D0) scattering,
classical solutions for various branes and fluctuations, etc

For a comprehensive review on this subject up until the early 2000s, see W. Taylor, hep-th/0101126 (published in Reviews of Modern Physics)

For myself,
a (relatively) more recent and suggestive piece of evidence:
consistency with "holographic" predictions
for ***non-perturbative*** 2-point correlators;

$$\Big\langle \mathcal{O}(t)\,\mathcal{O}(t')\Big\rangle \sim \frac{1}{g_s^2\,\ell_s^8}\frac{(g_s N \ell_s^7)^{1+\frac{2}{7}\nu}}{|t-t'|^{2\nu+1}}$$

$$\boxed{\nu = \frac{7}{5}\,(1-n_+ + n_-) + \frac{2}{5}\,\ell}$$

Y. Sekino and T.Y., Nucl. Phys. B570, 174(2000)[hep-th/990029]

- These **exponents** are intrinsically non-perturbative: they are independent of the coupling constant, but differ from the canonical dimensions of the operators;

  seems valid even for finite *N* !?

- The power-law behavior of "supergravity operators" reflects the **gapless** nature of this system.

suggestive "*experimental*" evidence

Using Monte Carlo simulations (*despite of a possible defect: sign problem ?*),

M. Hanada, J. Nishimura, Y.S. and T.Y., arXiv:0911.1623 (P.R.L), arXiv:1108.5135 (JHEP)

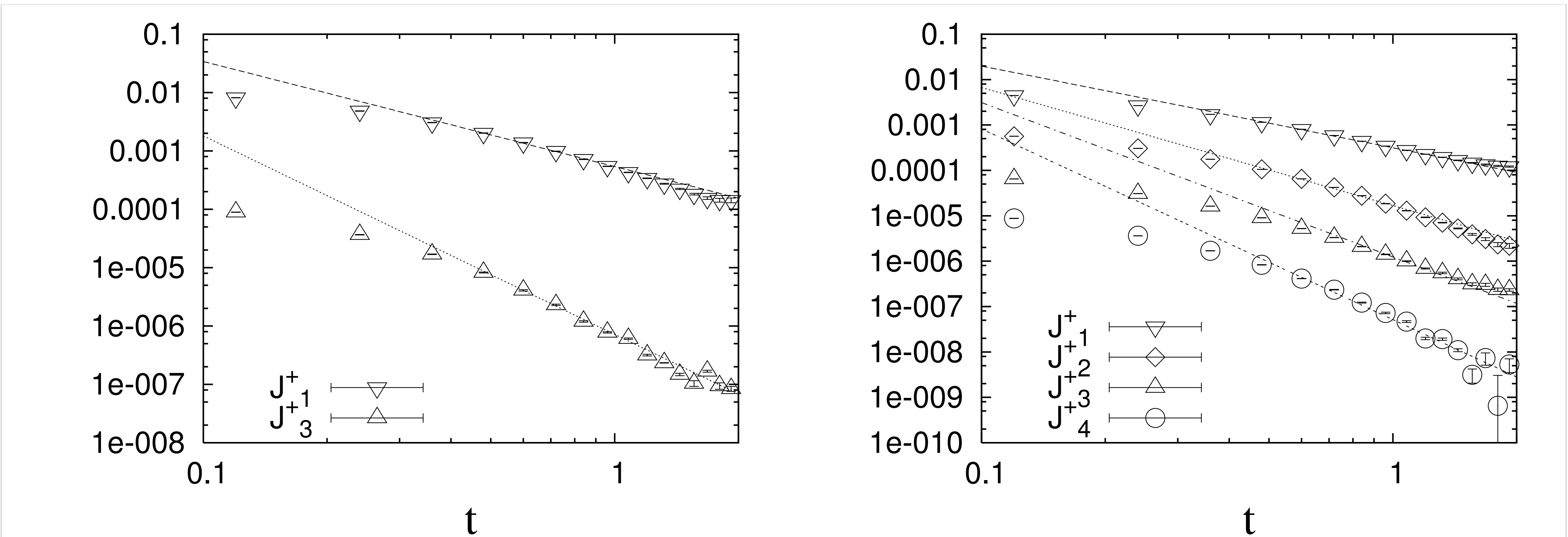


Figure 1. The log-log plot of the correlator $\left\langle J^+_\ell(t)\, J^+_\ell(0)\right\rangle$ with $\ell = 1, 3$ for $N = 2$ (Left) and with $\ell = 1, 2, 3, 4$ for $N = 3$ (Right). The cutoff parameters are chosen as $\beta = 4$ and $\Lambda = 16$. The straight lines represent the power-law behavior predicted by the gauge-gravity correspondence.

$$J^{+ij}_{\ell, i_1, \cdots, i_\ell} \equiv \frac{1}{N}\,\mathrm{Str}\Big(F_{ij} X_{i_1} \cdots X_{i_\ell}\Big) \qquad\qquad F_{ij} = -i\,[X_i, X_j]/\ell_s^2$$

Toward “covariantization” of DLCQ Matrix Theory

M(atrix)-theory proposal in the DLCQ scheme

BFSS conjecture (**IMF**)

$$H = \frac{1}{2P_{\circ}^{10}}(P_{\circ i}^{2} + \hat{H}), \qquad \hat{H} = N\mathrm{Tr}\Big(\hat{\boldsymbol{P}}_i^{2} - \frac{1}{2}[\boldsymbol{X}^i, \boldsymbol{X}^j]\Big) + \cdots$$

$i, j, \ldots,\ (1, 2, \ldots, 9)$ $\qquad P_{\circ}^{10} = N R_{11}^{-1} \to \infty$ $\qquad$ ($\circ$:center of mass)

D0s play the role of basic constituents
("***partons***") for all dynamical objects of M-theory

Susskind, Seiberg, Sen, ….. (1997)

The effective theory of ***N*** D-particles is re-interpreted as an ***exact* theory** in a particular light-front frame with a ***compactified light-like direction*** $\quad P_{\circ}^{+} = 2N/R$

$$-P_{\circ}^{-} = \frac{P_{\circ i}^{2} + M_{\mathrm{eff}}^{2}}{P_{\circ}^{+}}$$

by identifying $\hat{H}$ with M_{eff}^{2}

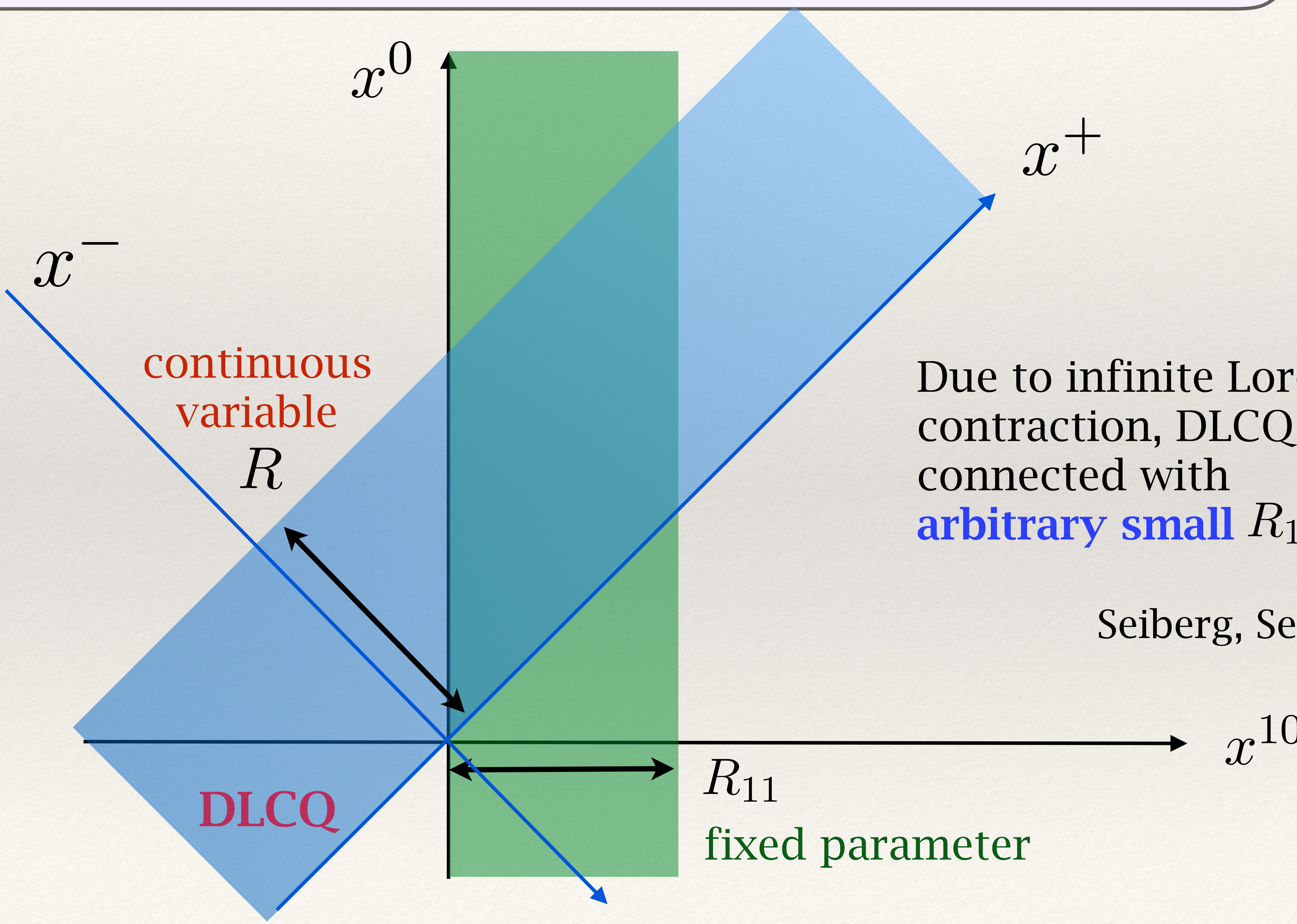
Boost transformation:

$$P_\circ^\pm \to P_\circ'^\pm = e^{\mp\rho} P_\circ^\pm$$

$$R' = e^\rho R$$

compactification radius

In spite of compactification, these are **continuously changing variables**



Due to infinite Lorentz contraction, DLCQ can be connected with **arbitrary small** R_{11}

Seiberg, Sen, …

Presumption for this hypothesis:

$\hat{H}$ **is physically equivalent with the** ***Lorentz-invariant*** M_{eff}^2 **even for finite and fixed** N **!**

Then, this must be true in arbitrary Lorentz frame, and hence, there should exist a Lorentz-covariant formulation of Matrix theory, such that we obtain $\hat{H}$ after appropriate gauge-fixing condition is imposed.

All of 11 dimensions should be treated on an equal footing, using 11 matrices in a new framework equipped with some ***higher gauge symmetries***.

remained unsolved for 20 years!

Higher gauge symmetry from Nambu bracket

Nambu bracket : a possible clue toward higher gauge symmetry ?

Nambu bracket naturally appears
in the classical theory of membrane

$$S = -\frac{1}{\ell_{11}^3}\int d^3\xi \sqrt{-\det\left(\frac{\partial X^\mu}{\partial \xi_a}\frac{\partial X_\mu}{\partial \xi_b}\right)} \quad \to \quad \frac{1}{2\ell_{11}^3}\int d^3\xi \left[\frac{1}{e}\{X^\mu, X^\nu, X^\sigma\}_{\mathrm{N}}^2 - e\right]$$

$$\{X^\mu, X^\nu, X^\sigma\}_{\mathrm{N}} \equiv \sum_{a,b,c} \epsilon^{abc}\partial_a X^\mu \partial_b X^\nu \partial_c X^\sigma$$

$\delta X^\mu = \{F, G, X^\mu\}_{\mathrm{N}}$ 3D vol.preserving diffeo. ($e = 1$ gauge)

In the light-front gauge, this reduces to 2D
area preserving diffeo.

$$\delta_F X^i = (\partial_1 F \partial_2 - \partial_2 F \partial_1) X^i = \{F, X^i\} \quad \Rightarrow \quad \delta_F \boldsymbol{X}^i = i[\boldsymbol{F}, \boldsymbol{X}^i]$$

What does correspond to $\delta X^\mu = \{F, G, X^\mu\}_\mathrm{N}$,"Nambu transformation", in terms of matrices?

We need something with an exact significance even for finite *N*!

We started this program in the late 90s:
Unfortunately, however, we could not accomplish it at that time.

Difficulty: ***causal time development*** should automatically emerge.
But how?

Our 1999 proposals of **finitely** discretized Nambu brackets :

Awata, Li, Minic and TY, "*On the quantization of Nambu bracket*",
hep-th/9906248; (JHEP 02(2001)013

(i) using **square** matrices, ***augmented by additional variables***

(ii) using cubic matrices with 3 indices A_{ijk}

We should seek finite matrix versions of Nambu bracket!

basic properties of classical Nambu bracket

1. skew symmetry $\{A_1, A_2, A_3\} = (-1)^{\epsilon(p)}\{A_{p(1)}, A_{p(2)}, A_{p(3)}\}$

2. Leipniz rule $\{A_1 A_2, A_3, A_4\} = A_1\{A_2, A_3, A_4\} + \{A_1, A_3, A_4\}A_2$

3. Jacobi-like ("fundamental") identity (FI)

$$\{A_1, A_2, \{A_3, A_4, A_5\}\} =$$
$$\{\{A_1, A_2, A_3\}, A_4, A_5\} + \{A_3, \{A_1, A_2, A_4\}, A_5\} + \{A_3, A_4, \{A_1, A_2, A_5\}\}$$

first introduced by Takhtajan, and others, ~1993

3. is the most crucial property from the viewpoint of symmetry transformations

We develop the approach (i) further, and extend it in the framework of *a Lorentz invariant* canonical formalism as a tool for realizing higher gauge symmetries.

T.Y., JHEP06(2016)058 [arXiv: 1603.06402]

*It should be emphasized that we do not assume the relation between the membrane action and Nambu bracket: such an analogy is **not essential** from the viewpoint of DLCQ interpretation. Matrix theory is **not just a regularization** of supermembrane.*

NB:

we are ***not*** going to pursue Nambu's original idea of "generalized Hamilton dynamics", in this review.

For another new development of Nambu dynamics from a different perspective (aiming towards "wave-mechanical quantization"), see T.Y., "Generalized Hamilton-Jacobi Theory of Nambu Mechanics", PTEP 023A01(2017)[arXiv:1612:08509].

Canonical formalism for higher gauge symmetries

(1) coordinate-type variables

- space-time vectors in 11 dimensions as functions of a Lorentz-invariant time-parameter τ

$$X^\mu(\tau) = (X^\mu_{\mathrm{M}}(\tau), \boldsymbol{X}^\mu(\tau))$$

"M"-variables
(auxiliary but dynamical)

$N \times N$ hermitian matrix variables

- **3-bracket**

$$[X, Y, Z] \equiv (0, X_{\mathrm{M}}[\boldsymbol{Y}, \boldsymbol{Z}] + Y_{\mathrm{M}}[\boldsymbol{Z}, \boldsymbol{X}] + Z_{\mathrm{M}}[\boldsymbol{X}, \boldsymbol{Y}])$$

◇ total skew symmetry

$$[X, Y, Z] = -[Y, X, Z] = -[X, Z, Y] = -[Z, Y, X]$$

◇ fundamental identity

$$[F, G, [X, Y, Z]] = [[F, G, X], Y, Z] + [X, [F, G, Y], Z] + [X, Y, [F, G, Z]]$$

local (with respect to τ) gauge transformation

$$\delta X^\mu = i[F, G, X^\mu]$$

$$\delta X^\mu_{\rm M} = 0$$

$$\delta \boldsymbol{X}^\mu = i[F_{\rm M}\boldsymbol{G} - G_{\rm M}\boldsymbol{F}, \boldsymbol{X}^\mu] + i[\boldsymbol{F}, \boldsymbol{G}]X^\mu_{\rm M}$$

generalized to $i\sum_r [F^r, G^r, X]$

$$\delta_{HL} X^\mu = \delta_H X^\mu + \delta_L X^\mu = (0, i[\boldsymbol{H}, \boldsymbol{X}^\mu]) + (0, \boldsymbol{L} X^\mu_{\rm M})$$

shift term

$$\boldsymbol{H} \equiv \sum_r F^r_{\rm M}\boldsymbol{G}^r - G^r_{\rm M}\boldsymbol{F}^r,$$

$$\boldsymbol{L} \equiv i\sum_r [\boldsymbol{F}^r, \boldsymbol{G}^r],$$

can be treated as two completely independent **traceless** matrices

◇ a single matrix can be gauged away to the unit matrix, if it is as sociated with ***non-zero M-variable***

$$X_\circ \equiv \frac{1}{N}{\rm Tr}(\boldsymbol{X}), \quad \boldsymbol{X} = X_\circ + \hat{\boldsymbol{X}}, \quad {\rm Tr}(\hat{\boldsymbol{X}}) = 0$$

center-of-mass coordinate

$$\hat{\boldsymbol{X}} \longrightarrow 0$$

◇ integral invariant (potential term)

$$\frac{1}{12}\int d\tau\, e\,\langle [X^\mu, X^\nu, X^\sigma][X_\mu, X_\nu, X_\sigma]\rangle$$

$$=\frac{1}{4}\int d\tau\, e\,\mathrm{Tr}\Big(X_{\mathrm{M}}^2[\boldsymbol{X}^\nu, \boldsymbol{X}^\sigma][\boldsymbol{X}_\nu, \boldsymbol{X}_\sigma] - 2[X_{\mathrm{M}}\cdot \boldsymbol{X}, \boldsymbol{X}^\nu][X_{\mathrm{M}}\cdot \boldsymbol{X}, \boldsymbol{X}_\nu]\Big)$$

$\boxed{\textit{einbein}}$ $d\tau e(\tau) = d\tau' e'(\tau')$

in order to preserve re-parametrization invariance

$$\langle [A,B,C],[X,Y,Z]\rangle \equiv$$

$$\mathrm{Tr}\Big((A_{\mathrm{M}}[\boldsymbol{B},\boldsymbol{C}] + B_{\mathrm{M}}[\boldsymbol{C},\boldsymbol{A}] + C_{\mathrm{M}}[\boldsymbol{A},\boldsymbol{B}])\,(X_{\mathrm{M}}[\boldsymbol{Y},\boldsymbol{Z}] + Y_{\mathrm{M}}[\boldsymbol{Z},\boldsymbol{X}] + Z_{\mathrm{M}}[\boldsymbol{X},\boldsymbol{Y}])\Big)$$

(2) momentum-type variables

canonical conjugates to the coordinate-type variables,
as independent dynamical variables

$$P^{\mu} = (P^{\mu}_{\mathrm{M}}, \boldsymbol{P}^{\mu})$$

"M-momentum"

$$\{X^{\mu}_{\mathrm{M}}, P^{\nu}_{\mathrm{M}}\}_{\mathrm{P}} = \eta^{\mu\nu},$$

$$\{X^{\mu}_{ab}, P^{\nu}_{cd}\}_{\mathrm{P}} = \delta_{ad}\delta_{bc}\eta^{\mu\nu},$$

$$\{X^{\mu}_{ab}, P^{\nu}_{\mathrm{M}}\}_{\mathrm{P}} = 0, \text{ etc}$$

Basic requirement: **canonical structure is preserved** under gauge transformations

$$\delta_{HL}\hat{\boldsymbol{P}}^{\mu} = i[\boldsymbol{H}, \hat{\boldsymbol{P}}^{\mu}] = \delta_{H}\boldsymbol{P}^{\mu},$$ ***no shift term***

$$\delta_{HL}P^{\mu}_{\mathrm{M}} = -\mathrm{Tr}\Big(\boldsymbol{L}\boldsymbol{P}^{\mu}\Big) = \delta_{L}P^{\mu}_{\mathrm{M}}$$

generator: $$\mathcal{C}_{HL} \equiv \mathrm{Tr}\Big(\boldsymbol{P}_{\mu}\big(i[\boldsymbol{H}, \boldsymbol{X}^{\mu}] + \boldsymbol{L}X^{\mu}_{\mathrm{M}}\big)\Big)$$

$$\delta_{HL}O = \{O, \mathcal{C}_{HL}\}_{\mathrm{P}}$$

two further requirements of symmetries

- **scale invariance** $\tau \rightarrow \lambda^2\tau$

$$\boldsymbol{X}^\mu \to \lambda \boldsymbol{X}^\mu, \quad X_{\mathrm{M}}^\mu \to \lambda^{-3} X_{\mathrm{M}}^\mu, \quad \boldsymbol{P}^\mu \to \lambda^{-1}\boldsymbol{P}^\mu, \quad P_{\mathrm{M}}^\mu \to \lambda^3 P_{\mathrm{M}}^\mu$$

- **gauge symmetry for eliminating negative modes associated with 11d Lorentz metric** $\eta^{\mu\nu}$

$$\delta_w X_\circ^\mu = w X_{\mathrm{M}}^\mu, \quad \delta_w P_\circ^\mu = 0, \quad \delta_w X_{\mathrm{M}}^\mu = 0, \quad \delta_w P_{\mathrm{M}}^\mu = -w P_\circ^\mu$$

$$\delta_Y \hat{\boldsymbol{X}}^\mu = 0, \quad \delta_Y \hat{\boldsymbol{P}}^\mu = P_\circ^\mu \boldsymbol{Y}, \quad \delta_Y X_\circ^\mu = -\mathrm{Tr}(\boldsymbol{Y}\hat{\boldsymbol{X}}^\mu), \quad \delta_Y P_\circ^\mu = 0$$

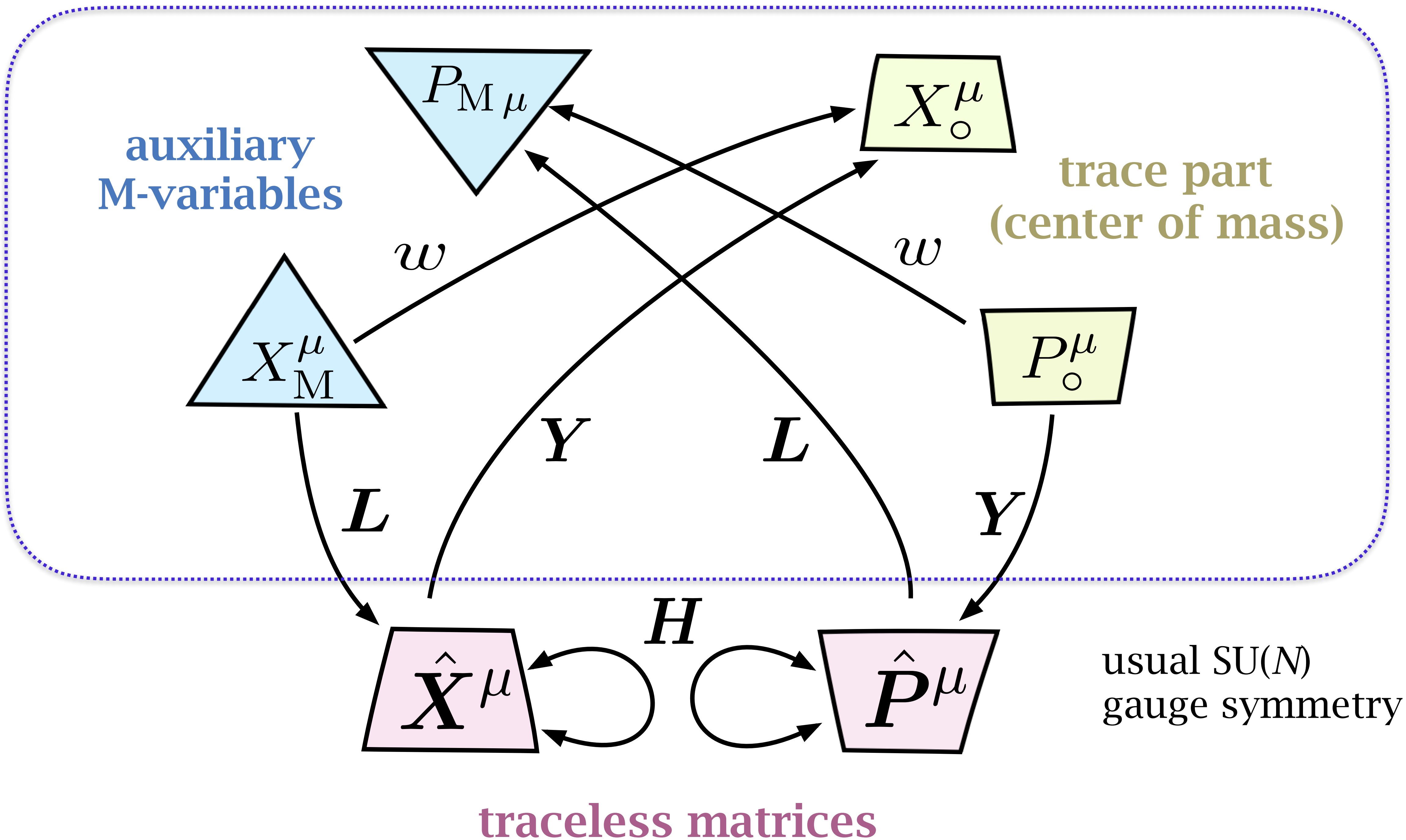
As a whole, the gauge symmetries are generated by

$$\mathcal{C}_{H+L+Y+w} = wP_\circ \cdot X_{\mathrm{M}} + \mathrm{Tr}\Big(-(P_\circ \cdot \boldsymbol{X})\boldsymbol{Y} + i\boldsymbol{P}_\mu[\boldsymbol{H}, \boldsymbol{X}^\mu] + (X_{\mathrm{M}} \cdot \boldsymbol{P})\boldsymbol{L}\Big)$$

$$\{\mathcal{M}^{\mu\nu}, \mathcal{C}_{HL+w+Y}\}_{\mathrm{P}} = 0$$

$$\mathcal{M}^{\mu\nu} \equiv X_{\mathrm{M}}^\mu P_{\mathrm{M}}^\nu - X_{\mathrm{M}}^\nu P_{\mathrm{M}}^\mu + \mathrm{Tr}(\boldsymbol{X}^\mu \boldsymbol{P}^\nu - \boldsymbol{X}^\nu \boldsymbol{P}^\mu)$$ Lorentz generator

Schematic structure of ***higher*** gauge symmetries



Covariant derivatives and generalized (Poincaré) integral invariant:

$$\int d\tau \left[P_{\mathrm{M}\,\mu} \frac{dX_{\mathrm{M}}^{\mu}}{d\tau} + \mathrm{Tr}\left(\boldsymbol{P}_{\mu} \frac{D\boldsymbol{X}^{\mu}}{D\tau} \right) \right]$$

$$= \int d\tau \left[P_{\mathrm{M}\,\mu} \frac{dX_{\mathrm{M}}^{\mu}}{d\tau} + P_{\circ\,\mu} \frac{DX_{\circ}^{\mu}}{D\tau} + \mathrm{Tr}\left(\hat{\boldsymbol{P}}_{\mu} \frac{D\hat{\boldsymbol{X}}^{\mu}}{D\tau} \right) \right]$$

$$\frac{DX_{\circ}^{\mu}}{D\tau} = \frac{dX_{\circ}^{\mu}}{d\tau} - eBX_{\mathrm{M}}^{\mu} + e\mathrm{Tr}(\boldsymbol{Z}\hat{\boldsymbol{X}}^{\mu})$$

$$\frac{D\hat{\boldsymbol{X}}^{\mu}}{D\tau} = \frac{d\hat{\boldsymbol{X}}^{\mu}}{d\tau} + ie[\boldsymbol{A}, \boldsymbol{X}^{\mu}] - e\boldsymbol{B}X_{\mathrm{M}}^{\mu}$$

contributions from gauge fields

symplectic structure

such that it is invariant under generalized (and finitely discretized) Nambu transformations.

11d covariant action of Matrix theory

Lorentz invariant action (bosonic part)

$$A_{\rm boson} = \int d\tau \Big[P_\circ \cdot \frac{DX_\circ}{D\tau} + P_{\rm M} \cdot \frac{dX_{\rm M}}{d\tau} + {\rm Tr}\Big(\hat{\boldsymbol{P}} \cdot \frac{D\hat{\boldsymbol{X}}}{D\tau}\Big)$$

$$- \frac{e}{2N} P_\circ^2 - \frac{e}{2} {\rm Tr}(\hat{\boldsymbol{P}} - P_\circ \boldsymbol{K})^2 + \frac{e}{12} \langle [X^\mu, X^\nu, X^\sigma][X_\mu, X_\nu, X_\sigma] \rangle \Big]$$

$\delta_Y \boldsymbol{K} = \boldsymbol{Y}$ additional (auxiliary) Stückelberg variable

Or, if you like, we can set $\boldsymbol{K} = (P_\circ^2)^{-1} \hat{\boldsymbol{P}} \cdot P_\circ$

(1) Local reparametrization invariance with respect to τ

(2) Global translation invariance

$$X_\circ^\mu \to X_\circ^\mu + c^\mu \qquad P_{\rm M}^\mu \to P_{\rm M}^\mu + b^\mu \Rightarrow \frac{dP_\circ^\mu}{d\tau} = 0 \qquad \frac{dX_{\rm M}^\mu}{d\tau} = 0$$

(3) Global scaling symmetry $\quad \tau \longrightarrow \lambda^2 \tau$

$$\boldsymbol{X}^\mu \to \lambda \boldsymbol{X}^\mu, \quad X_{\rm M}^\mu \to \lambda^{-3} X_{\rm M}^\mu, \quad \boldsymbol{P}^\mu \to \lambda^{-1} \boldsymbol{P}^\mu, \quad P_{\rm M}^\mu \to \lambda^3 P_{\rm M}^\mu$$

(4) Gauge symmetries under $\delta_H + \delta_L + \delta_Y + \delta_w$

Mass-shell condition and Gauss constraints:

einbein $\Rightarrow$ $P_\circ^2 + \mathcal{M}_{\text{boson}}^2 \approx 0$

$$\mathcal{M}_{\text{boson}}^2 = N\mathrm{Tr}(\hat{\boldsymbol{P}} - P_\circ \boldsymbol{K})^2 - \frac{N}{6}\langle [X^\mu, X^\nu, X^\sigma][X_\mu, X_\nu, X_\sigma] \rangle$$

Gauss constraints (satisfying a closed algebra)

gauge fields

traceless matrices

$\boldsymbol{A}$ $\quad [\boldsymbol{P}_\mu, \boldsymbol{X}^\mu] + \cdots \approx 0$

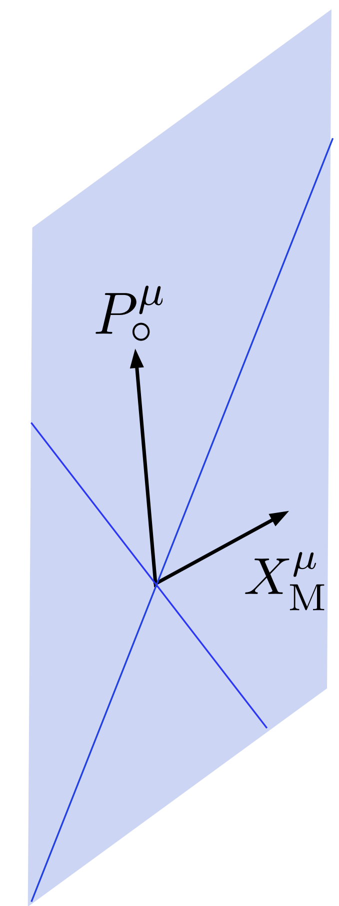
$\boldsymbol{B}$ $\Rightarrow$ $\hat{\boldsymbol{P}}_\mu X_{\mathrm{M}}^\mu \approx 0$

$\boldsymbol{Z}$ $\quad \hat{\boldsymbol{X}}_\mu P_\circ^\mu \approx 0$

B $\Rightarrow$ $P_{\circ\,\mu} X_{\mathrm{M}}^\mu \approx 0$

time-like space-like

both conserved



"M-plane" or "M-frame", replacing c.m.frame

Light-front gauge fixing:

choose a Lorentz frame such the M-plane coincides with “10-0” plane

$$P_\circ^\pm \equiv P_\circ^{10} \pm P_\circ^0, X_{\rm M}^\pm \equiv X_{\rm M}^{10} \pm X_{\rm M}^0$$

δ_L-transformation $\Rightarrow$ $\hat{\boldsymbol{X}}^+ = 0$

Z-gauss constraint:

$$0 = P_\circ^+ \hat{\boldsymbol{X}}^- + P_\circ^- \hat{\boldsymbol{X}}^+ = P_\circ \hat{\boldsymbol{X}}^- \quad \Rightarrow \hat{\boldsymbol{X}}^- = 0$$

eq. of motion: $\boldsymbol{K} = 0$ gauge with respect to δ_Y

$$\hat{\boldsymbol{P}}^\pm = \frac{1}{e}\frac{d\hat{\boldsymbol{X}}^\pm}{d\tau} + i[\boldsymbol{A}, \hat{\boldsymbol{X}}^\pm] - \boldsymbol{B} X_{\rm M}^\pm \quad \Rightarrow \boldsymbol{P}^\pm = -\boldsymbol{B} X_{\rm M}^\pm$$

B-gauss constraint:

$$0 = X_{\rm M} \cdot \hat{\boldsymbol{P}} \quad \Rightarrow \quad \boldsymbol{B} X_{\rm M}^2 = 0 \quad \boldsymbol{B} = 0 \quad \Rightarrow \hat{\boldsymbol{P}}^\pm = 0$$

Light-like components of ***traceless matrix variables*** are completely eliminated, due to the Gauss constraints of higher gauge symmetries

$$\mathcal{M}^2_{\rm boson} = N\mathrm{Tr}\Big(\hat{\boldsymbol{P}}_i^2 - \frac{1}{2}X_{\rm M}^2[\boldsymbol{X}_i, \boldsymbol{X}_j]^2\Big) = \hat{H}$$

$$[\boldsymbol{X}_i, \boldsymbol{P}_i] = 0$$

$$P^{\pm}_{\circ} = N\frac{dX^{\pm}_{\circ}}{ds} \quad \Rightarrow \quad X^{+}_{\circ} = \frac{P^{+}_{\circ}}{N}s \quad \Rightarrow \quad P^{+}_{\circ} = \frac{2N}{R}$$

$$ds = e d\tau$$

constant parameter, independent of N

$$\mathrm{SU}(N_1) \subset \mathrm{SU}(N_1)\times \mathrm{SU}(N_2) \subset \mathrm{SU}(N_1+N_2)$$

all subsystems are synchronized with a single (common) proper time, irrespective of N

DLCQ compactification condition is an automatic consequence of synchronization with a single and invariant time parameter

If we adopt the BFSS condition for compactification $P_\circ^{10} = \dfrac{N}{R_{11}}$

the system ends up with a **Born-Infeld-like effective action**

$$A_{\text{spat boson}} = \int dt \Big[\mathrm{Tr}\Big(\hat{\boldsymbol{P}}_i \frac{D\hat{\boldsymbol{X}}_i}{Dt} \Big) - P_\circ^0 \Big]$$

$$P_\circ^0 = \sqrt{(P_\circ^{10})^2 + N\mathrm{Tr}\Big(\hat{\boldsymbol{P}}_i \cdot \hat{\boldsymbol{P}}_i - \frac{1}{2} X_{\mathrm{M}}^2 [\boldsymbol{X}_i, \boldsymbol{X}_j][\boldsymbol{X}_i, \boldsymbol{X}_j] \Big)}$$

$$\Rightarrow \qquad -\int dt\, \mathcal{M}_{\text{spat}} \sqrt{N} \Big[1 - \frac{1}{N} \mathrm{Tr}\Big(\frac{D\hat{\boldsymbol{X}}_i}{Dt} \frac{D\hat{\boldsymbol{X}}_i}{Dt} \Big) \Big]^{1/2}$$

$$\mathcal{M}_{\text{spat}} \equiv \Big[\frac{N}{R_{11}^2} - \frac{1}{2\ell_{11}^6} \mathrm{Tr}\big([\boldsymbol{X}_i, \boldsymbol{X}_j][\boldsymbol{X}_i, \boldsymbol{X}_j] \big) \Big]^{1/2}$$

$$\Rightarrow \int dt\, \frac{N}{R_{11}} \Big[-1 + \frac{1}{2N} \mathrm{Tr}\Big(\frac{D\hat{\boldsymbol{X}}_i}{Dt} \frac{D\hat{\boldsymbol{X}}_i}{Dt} + \frac{R_{11}^2}{2\ell_{11}^6} [\boldsymbol{X}_i, \boldsymbol{X}_j][\boldsymbol{X}_i, \boldsymbol{X}_j] \Big) + O\big(\frac{1}{N^2}\big) \Big]$$

consistent with the original BFSS conjecture

Other roles of the "M-variables"

- Emergence of the fundamental scale of M-theory:

"spontaneous breaking" of global scaling symmetry

(at initial time of the universe)

(or **a super selection rule)**

$$X_{\mathrm{M}}^2 = \frac{1}{\ell_{11}^6}$$

- Covariant formulation of ***(dynamical) supersymmetry***

by providing a natural covariant projection condition on spinor matrices $\boldsymbol{\Theta}$: $P_- \boldsymbol{\Theta} = \boldsymbol{\Theta}, \ P_+ \boldsymbol{\Theta} = 0$

$$P_\pm \equiv \frac{1}{2}(1 \pm \Gamma_\circ \Gamma_{\mathrm{M}}) \qquad \Gamma_{\mathrm{M}} \equiv \frac{X_{\mathrm{M}} \cdot \Gamma}{\sqrt{X_{\mathrm{M}}^2}}, \quad \Gamma_\circ \equiv \frac{P_\circ \cdot \Gamma}{\sqrt{-P_\circ^2}}$$

total action: $$A = A_{\mathrm{boson}} + A_{\mathrm{fermion}}$$

$$A_{\mathrm{fermion}} = \int d\tau \left[\bar{\Theta}_\circ P_\circ \cdot \Gamma \frac{d\Theta_\circ}{d\tau} + \frac{1}{2} \mathrm{Tr} \Big(\bar{\boldsymbol{\Theta}} \Gamma_\circ \frac{D\boldsymbol{\Theta}}{D\tau} \Big) - e \frac{i}{4} \langle \bar{\Theta}, \Gamma_{\mu\nu} [X^\mu, X^\nu, \Theta] \rangle \right]$$

Supersymmetry

Global (kinematical) susy: $\delta_\varepsilon X_\circ^\mu = \bar{\varepsilon}\Gamma^\mu\Theta_\circ, \qquad \delta_\varepsilon\Theta_\circ = -\varepsilon$

Generic states obey the massive representation of dimension 2^{16} , being many body states composed of 11d basic graviton multiplets.

Internal (dynamical) susy, in each sector of conserved $P_\circ^\mu$ and X_{M}^μ

$$\delta_\epsilon \hat{\boldsymbol{X}}^\mu = \bar{\epsilon}\Gamma^\mu\boldsymbol{\Theta},$$

$$\delta_\epsilon \hat{\boldsymbol{P}}_\mu = i\sqrt{X_{\mathrm{M}}^2}\,[\bar{\boldsymbol{\Theta}}\Gamma_{\mu\nu}\epsilon, \tilde{\boldsymbol{X}}^\nu], \quad \delta_\epsilon\boldsymbol{K} = 0,$$

$$\delta_\epsilon\boldsymbol{\Theta} = P_-\big(\Gamma_\circ\Gamma_\mu\hat{\boldsymbol{P}}^\mu\epsilon - \frac{i}{2}\sqrt{X_{\mathrm{M}}^2}\,\Gamma_\circ\Gamma_{\mu\nu}\epsilon[\tilde{\boldsymbol{X}}^\mu, \tilde{\boldsymbol{X}}^\nu]\big),$$

$$\delta_\epsilon\boldsymbol{A} = \sqrt{X_{\mathrm{M}}^2}\,\bar{\boldsymbol{\Theta}}\epsilon,$$

$$\delta_\epsilon\boldsymbol{B} = i\big(X_{\mathrm{M}}^2\big)^{-1}[\delta_\epsilon\boldsymbol{A}, X_{\mathrm{M}}\cdot\boldsymbol{X}],$$

$$\delta_\epsilon\boldsymbol{Z} = i(P_\circ^2)^{-1}[\delta_\epsilon\boldsymbol{A}, P_\circ\cdot\boldsymbol{P}] + \frac{X_{\mathrm{M}}^2}{2P_\circ^2}([\delta_\epsilon\boldsymbol{X}^\mu, [P_\circ\cdot\boldsymbol{X}, \boldsymbol{X}_\mu]] + [\boldsymbol{X}^\mu, [P_\circ\cdot\boldsymbol{X}, \delta_\epsilon\boldsymbol{X}_\mu]])$$

$$\tilde{\boldsymbol{X}}^\mu = \boldsymbol{X}^\mu - \frac{1}{X_{\mathrm{M}}^2}X_{\mathrm{M}}^\mu(\boldsymbol{X}\cdot X_{\mathrm{M}}) - \frac{1}{P_\circ^2}P_\circ^\mu(\boldsymbol{X}\cdot P_\circ)$$

- It is a straightforward task to quantize this system in a manifestly covariant fashion, **including fermions, thanks to the presence of the M-variables**, using, say, the BFV formalism with propagating gauge fields which enable us to **eliminate all the Gauss (and second-class) constraints**.

- ***What about non-perturbative aspects ?***

The situation may be somewhat different from ordinary local field theories; the Matrix theory is intrinsically a ***configuration-space and non-local formulation*** *of many-body systems of stable D0 branes.*

In a sense, "Vacuum" is assumed to be trivial, no vacuum polarization. There is, however, some subtlety in the limit of $P_\circ^2 \to 0$ *. But the spectrum is continuous, we can always add soft gravitons such that we have states with non-vanishing squared mass.*

Concluding Discussions

Remaining problems and future prospects

Hopefully, the present covariant re-formulation of Matrix theory would be useful as an intermediate step toward M-theory.

Many issues remain to be clarified:

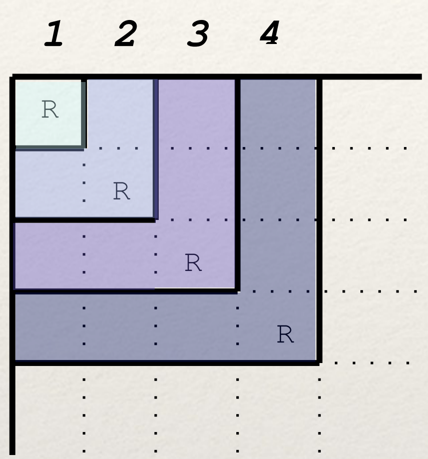
- dynamics (& dualities) of various branes in the large N
- background dependence and/or independence
- correspondence with 11d supergravity
- covariantization of (& connection to) Matrix-string theory
- connection to ABJM, …. and also to type IIB,….
- inclusion of anti-D0 branes, SSB of supersymmetires

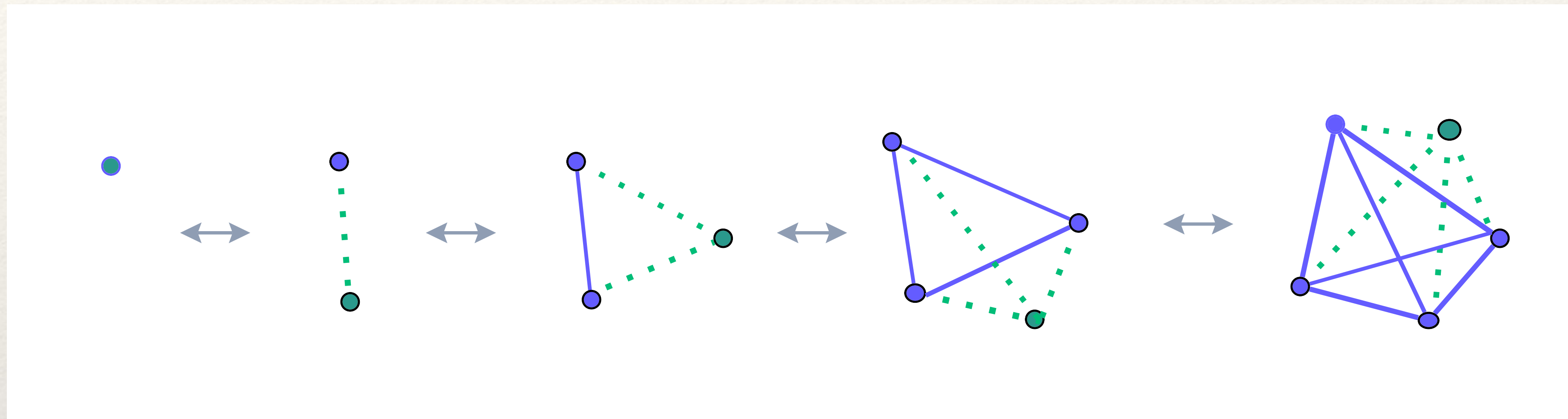
and so forth…..

The last issue actually suggests the limitation of the present Matrix-theory approach !
It describes only a special sector of M-theory.

Ultimately, we should develop something like
***field theory of D-branes** (and anti-D-branes)*

Introduce a field operator that allows us to go back and forth among Hilbert spaces with all the different *N*s in the Fock space of D-branes

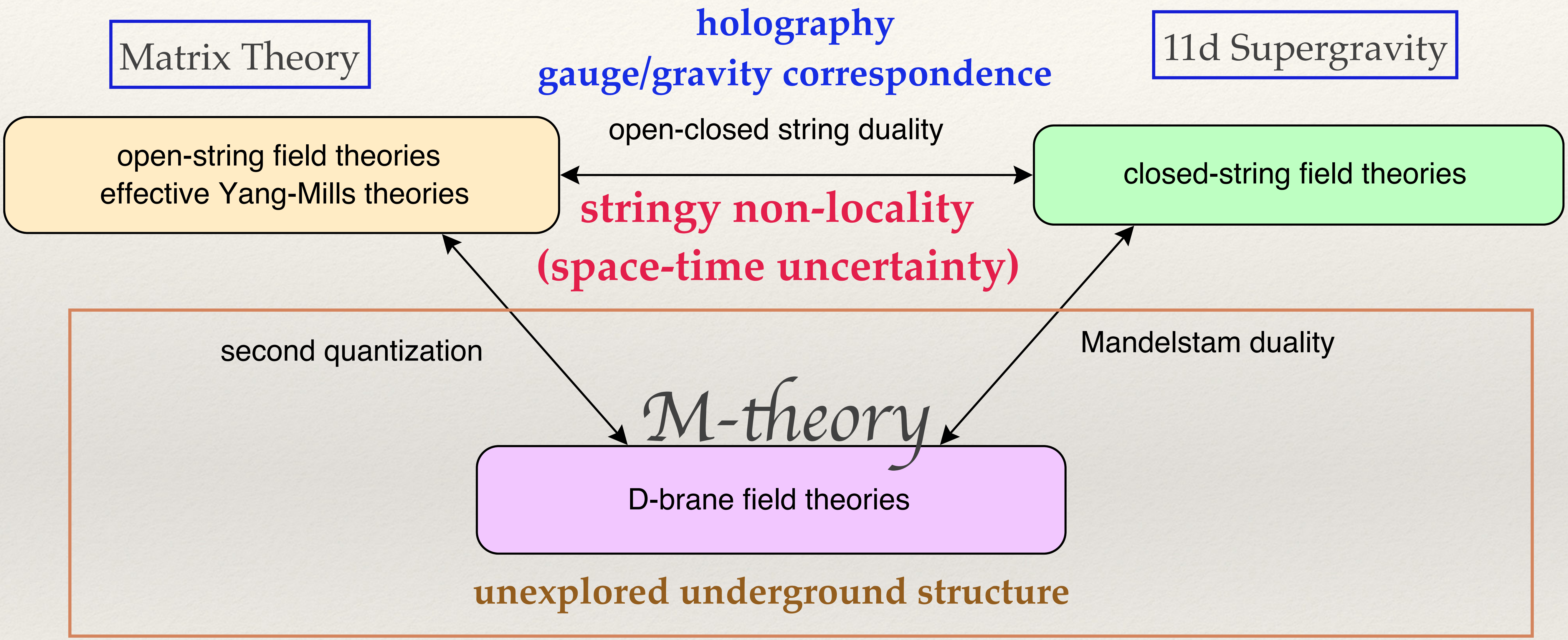




We will end up with a peculiar non-local field theory on
an infinite dimensional complex vector (super) space
as the base space.

Physical objects = bi-linears of the fields ("currents")

For preliminary attempts toward such a direction,
T.Y., PTP. 118, 135(2007)[arXiv:0705.1960, also 0804.0297];
also JHEP 12(2005)028[arXiv:hep-th/0510114].



Matrix Theory
holography
gauge/gravity correspondence
11d Supergravity
open-string field theories
effective Yang-Mills theories
open-closed string duality
closed-string field theories
stringy non-locality
(space-time uncertainty)
second quantization
Mandelstam duality
M-theory
D-brane field theories
unexplored underground structure

Epilogue

Remarks on the s*ignificance of string/M theory from a historical perspective in the long run*

International Conference of Theoretical Physics, Kyoto & Tokyo (1953)

R. P. Feynman,

responding to **S. Sakata**'s question about general (or "philosophical") ***status of field theories***, in session "Field Theory A. **Non-Local Theory**", chaired by A.Pais, "I would like to say why I think there is some interest in the **non-local field theories**, because they have been demolished temporarily. The difficulty of obtaining a non-local field theory is amazing. I ask "Why is it so difficult?" If we take the principle of relativity, and the principle of superposition of amplitudes, that is quantum mechanics, and put them together, we can not contain in the system an arbitrary function. Now non-relativistically we can put in any potential, but relativistically we can not.

Only if we get some **crazy** theory, (it does not make any difference what) but some **consistent** one **which is able to do that**, will we discover perhaps some fundamental idea has changed. Maybe that would give a **clue about what it is**.

So one reason why the non-local theory attempts are interesting is to try to find out why it is so hard to do."

Einstein had said something similar:

> "So long as no one has new concepts, which appear to have sufficient constructive power, mere doubt remains; that is, unfortunately, my own situation. Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything organic to take place. "

in *Reply to Criticisms, 1949*

comment on a proposal by mathematician K. Menger, that, for geometrization of the physics of the microcosm, one alternative to smooth Riemann spaces is a geometry where points are not primary entities, or a theory in which lumps are undefined concepts, whereas points appear as the results of limiting or intersectional process applied to these lumps.

(*reminiscent of Yukawa's proposal, "elementary domains", in the late 60th*)

Of course, through the development of string theory from the early 1970s to the present, we have had incomparably richer experiences and deeper insights into non-local field theories than the early 1950s.

In my opinion,
String/M theory
(together with associated matrix models)
is indeed a "**crazy**" *but* "**consistent**" theory,
"**which is able to do that**" **and has** "**sufficient constructive power.**"
Surely, it gives "**a clue about what it is**".

Conjecture:
the non-locality of string theory should be unified with the non-locality of a different kind, that is intrinsic to the general concept of quantum mechanical states (through quantum entanglements?).

It seems unfortunate that, in recent years, new development of string/M theory proper seems rather scarce.
But I hope that people would come back to the real issues of string/M theory, in the not-so distant future.

Do we have really "fundamental ideas"?

Do we have clues to experimental verification?

Thanks !