超弦理論における因果律と行列模型

Yuhma Asano (University of Tsukuba)
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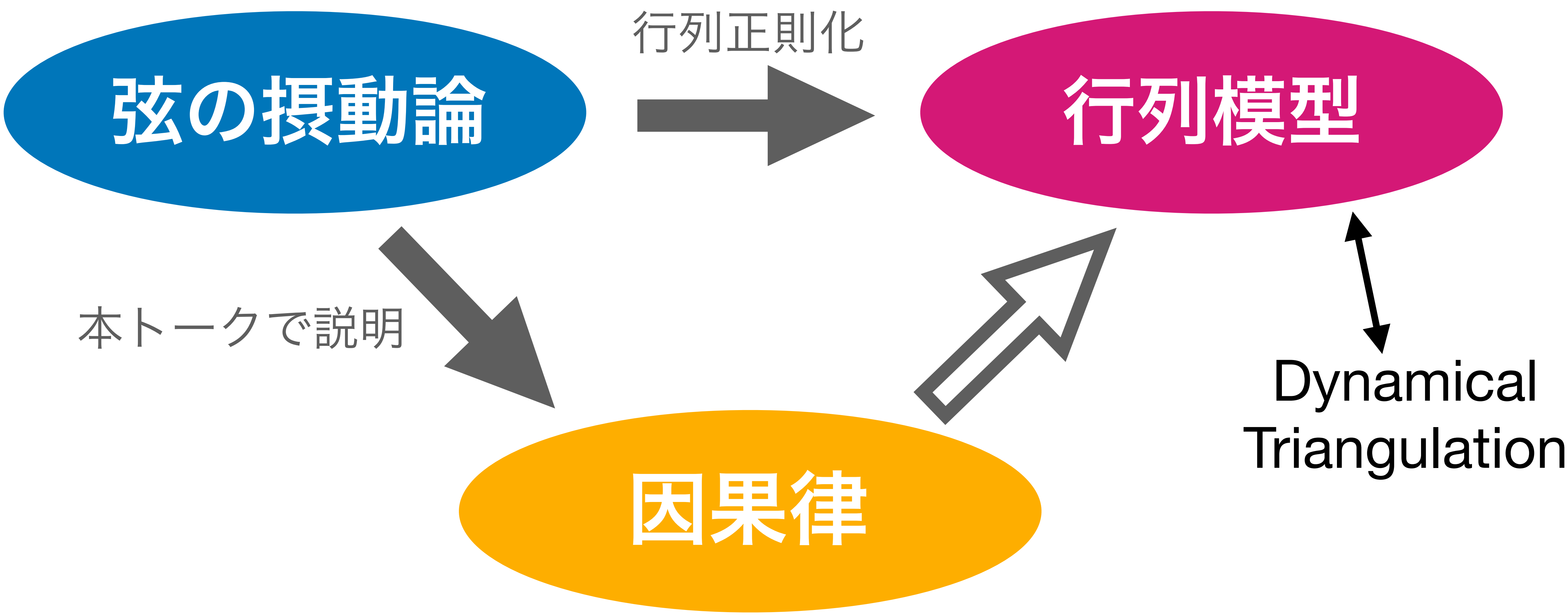
Overview

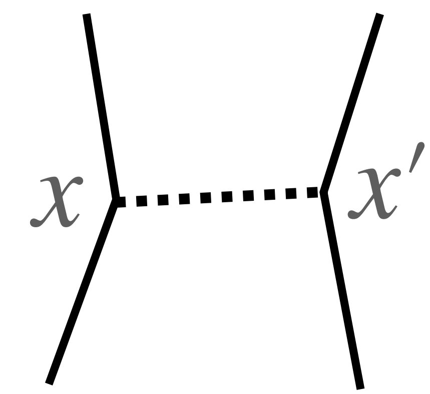


行列模型における因果律とは何か？

逆に、弦理論で実現される因果律を通して
“因果的”行列模型を導き出したい。

Overview

因果律



普通のQFTでの因果律:　$[\mathscr{H}_{\rm int}(x), \mathscr{H}_{\rm int}(x')] = 0$

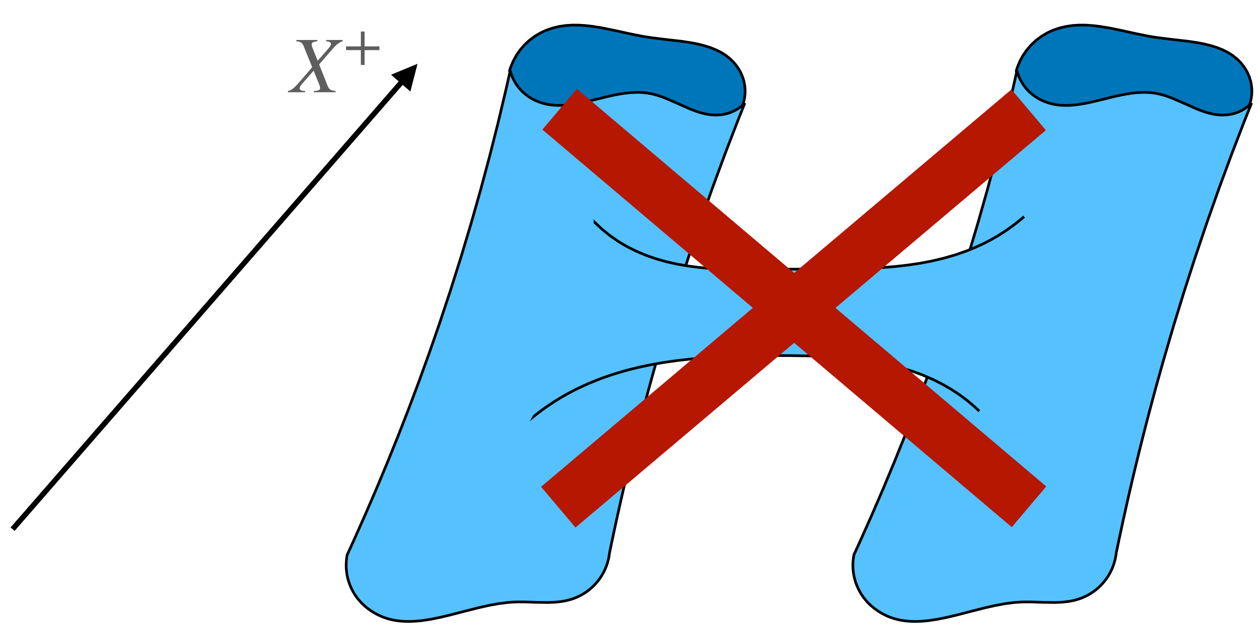
これは反粒子の存在によって実現されていた。

$[\phi(x), \phi^\dagger(x')]_\mp = 0$　　for space-like separation $(x-x')^2 > 0$

摂動的弦理論では、

因果律を実現する明らかな方法がなさそうにみえる。　cf. [Witten '13]

素朴にはspace-likeな“伝播”が禁止されることが期待される。



Overview

Matrix model

Matrix model is proposed as a non-perturbative formulation.

- c=1 matrix model: 1D matrix Q.M. (bosonic)

  ⋯ 2D bosonic/0B string theory

- BFSS model: 1D matrix Q.M. w/ SUSY

  ⋯ DLCQ M-theory

- IKKT model: 0D matrix model w/ SUSY

  ⋯ type IIB string theory

E.g. One-matrix model (0D matrix model)

$$S_{1M} = N \operatorname{tr}\left(\frac{1}{2}M^2 + \frac{\lambda}{4}M^4\right)$$

$N \to \infty$ with
$g_s = N^{-1}(\lambda + \frac{1}{12})^{-5/4}$ fixed

correlator:

$$A(N,\lambda) = A_0 + \frac{(\lambda+\frac{1}{12})^{-\frac{5}{2}}A_1}{N^2} + \frac{(\lambda+\frac{1}{12})^{-5}A_2}{N^4} + \cdots$$

$$\to A_{\mathrm{DSL}}(g_s) = \sum_{h=0}^{\infty} g_s^{2h}\left(A_h + e^{-\alpha/g_s}A_h^{(1)} + \cdots\right)$$

This model describes 2D pure gravity/1D critical bosonic string

Overview

The IKKT matrix model

[Ishibashi, Kawai, Kitazawa, Tsuchiya '96]

$$S[X,\psi] = N\,\mathrm{tr}\left[\frac{1}{4}[X^\mu, X^\nu][X_\mu, X_\nu] + \frac{1}{2}\psi^T\Gamma^\mu[X_\mu, \psi]\right]$$

X^μ: bosonic $N \times N$ matrices ($\mu = 0,\cdots,9$) ψ: Majorana-Weyl fermionic $N \times N$ matrices

This 0-dimensional theory is considered to describe type IIB superstring theory non-perturbatively. We believe this because it:

- has supersymmetry identical to that of type IIB string: $\mathcal{N} = (2,0)$ in (9+1)D
- reproduces perturbative results
  (graviton-exchange potential, D-brane scattering amplitudes, etc.)
- can reproduce the light-cone string field theory by the Schwinger-Dyson eq.
  [Fukuma, Kawai, Kitazawa, Tsuchiya '97]
- has potential to dynamically realise (3+1)D space-time at large N
  - Dynamics of the diagonal elements of X^μ forms 4D [Aoki, Iso, Kawai, Kitazawa, Tada '98]
  - SSB to SO(3) is observed [Anagnostopoulos, Azuma, Ito, Nishimura, Okubo, Papadoudis '20; Kumar, Joseph, Kumar '22]

Overview

Problem: How is the 0D theory defined?

The IKKT action:

$$S[X,\psi;G_{\mu\nu}] = N\,\mathrm{tr}\left[\frac{1}{4}G_{\mu\rho}G_{\nu\sigma}[X^\mu,X^\nu][X^\rho,X^\sigma] + \frac{1}{2}\psi^T G_{\mu\nu}\Gamma^\mu[X^\nu,\psi]\right]$$

However, we don't really know how the IKKT action enters in the partition fn.

$$Z = \int [dX][d\psi]\, e^{iS[X,\psi;\eta_{\mu\nu}]} \;?$$

$$\left(\eta_{\mu\nu} = \mathrm{diag}(-1,1,\cdots,1)_{\mu\nu}\right)$$

		metric in the action	
		Euclidean	Minkowski
weight	Euclidean	$e^{-S[X,\psi;\delta_{\mu\nu}]}$	$e^{-S[X,\psi;\eta_{\mu\nu}]}$
	Minkowski	$e^{iS[X,\psi;\delta_{\mu\nu}]}$	$e^{iS[X,\psi;\eta_{\mu\nu}]}$

"Euclidean IKKT model"

(Lorentzian)
"Minkowskian IKKT model"

We will try to answer this by revisiting perturbative string theory.

Overview

Ambiguity of the Minkowskian IKKT model

The Minkowskian IKKT w/ e^{iS} w/o regulators is conditionally convergent.

→ There are various definitions depending on how to make it finite.

E.g.

1. Mass term + Lorentz-sym. breaking cutoff

$$S = N\,\mathrm{tr}\left[\frac{1}{4}[X^\mu, X^\nu]^2 + \gamma\,\mathrm{tr}(e^{i\epsilon}X^i X^i - e^{-i\epsilon}X^0 X^0)\right]$$

- $\gamma \to 0^-$: equiv. to the Euclidean IKKT $\left(X^i = e^{-\frac{i}{4}\theta}\tilde{X}^i,\ X^0 = e^{\frac{3i}{4}\theta}\tilde{X}^{10},\quad \theta:\ 0 \to \frac{\pi}{2}\right)$
- $\gamma \to 0^+$: a different theory

[Y.A., Nishimura, Piensuk, Yamamori, to appear]

2. Lorentz symmetry "gauge-fixed" model

$$Z = \int DX\,D\psi\,\Delta_{\mathrm{FP}}[X]\,\prod_{i=1}^{9}\delta(\mathrm{tr}(X^0X^i))\,e^{iS[X,\psi;\eta_{\mu\nu}]}$$

[Y.A., Nishimura, Piensuk, Yamamori '24; Chou, Nishimura, Tripathi '25]

We do not discuss such ambiguity in this talk.

Overview

“Causal” matrix model

Matrix regularisation of type IIB superstring: IKKT matrix model

$$S_{\mathrm{IKKT}} = N\,\mathrm{tr}\left[\frac{1}{4}[X^\mu, X^\nu][X_\mu, X_\nu] + \frac{1}{2}\psi^T\Gamma^\mu[X_\mu, \psi]\right]$$

If we change how we apply the regularisation,

$$S_{\mathrm{NBI}} = N\,\mathrm{tr}\left[\frac{1}{4}Y^{-1}[X^\mu, X^\nu]^2 + \frac{1}{2}\psi^T\Gamma^\mu[X_\mu, \psi] + Y + \frac{i}{N}(N+\tfrac{1}{2})\ln(-iY)\right]$$

Y: bosonic $N \times N$ matrix

This is a Minkowski ver. of the NBI matrix model and “causal”.

[Fayyazuddin, Makeenko, Olesen, Smith, Zarembo ’96]

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2. Classical Perturbative String
   Equivalence between the 3 types of formulation

3. Quantum Perturbative String

4. “Derivation” of the Matrix Model

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Introduction

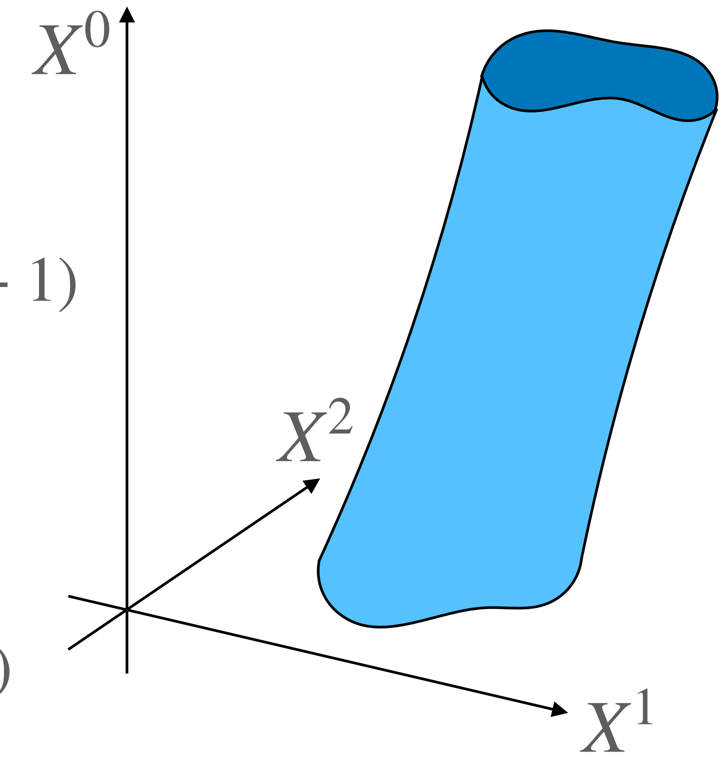
Definition of string theory

We usually start with the Nambu-Goto action:

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int d\sigma^0 d\sigma^1 \sqrt{-\det \partial_a X_\mu \partial_b X^\mu} \quad (\mu = 0,\cdots,D-1)$$

$$S_{\mathrm{P}}^{(\mathrm{E})} = \frac{1}{4\pi\alpha'}\int d\sigma^1 d\sigma^2 \sqrt{g}\, g^{ab}\partial_a X_m \partial_b X^m$$

Euclidean Polyakov-type action $(m = 1,\cdots,D)$



An S-matrix is defined by

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi}\int DX\, Dg\; V_{j_1}(k_1)\cdots V_{j_n}(k_n)\; \exp[-S_{\mathrm{P}}^{(\mathrm{E})}]$$

[Polyakov '81]

$\cdots$ but this is just perturbative expansion and no non-perturbative information is included.

E.g. the true vacuum cannot be determined.

Introduction

Perturbative string theory

(Euclidean)
$S_{\rm P}^{\rm (E)}$: Polyakov-type action

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\, D\theta\, Dg\, V_{j_1}(k_1)\cdots V_{j_n}(k_n)\ \exp[-S_{\rm P}^{\rm (E)}]$$

Questions:

- Is it equivalent to the Minkowskian theory?
- Is it equivalent to the Nambu-Goto-type formulation?
- For a Minkowskian theory, how do we define the Nambu-Goto-type formulation in a path integral?

$$\exp\left[-i\int d^2\sigma\sqrt{-\det h_{ab}}\right]$$

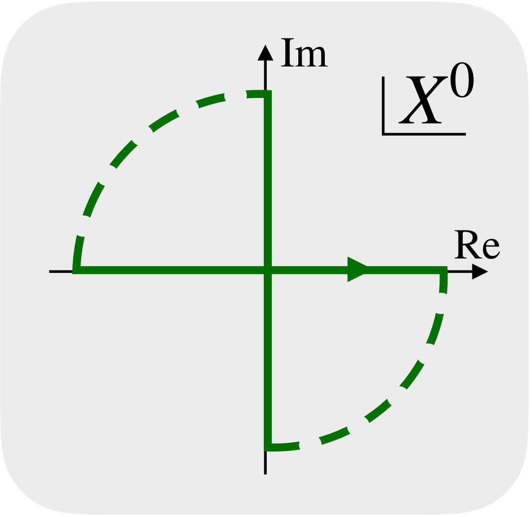
- Does it have the same features as standard QFT? Causality?

Introduction

Euclidean v. Minkowskian

We start with the Minkowski signature but at some point, Wick-rotate the theory to the Euclidean signature.

⋯ Because Euclidean theory is usually well-defined



But we should NOT naively Wick-rotate it; otherwise, we might arrive at a different theory.

$$X^0 = e^{-i\theta} X^D, \qquad e_0{}^a = e^{i\theta} e_2{}^a \qquad \left(\theta : 0 \to \frac{\pi}{2}\right)$$

$$\int DX\, Dg\, \exp\left[-i \int d^2\sigma \sqrt{-g}\, g^{ab} \partial_a X^\mu \partial_b X_\mu\right] \qquad g^{ab} = e_\alpha{}^a \eta^{\alpha\beta} e_\beta{}^b$$

$$= \int DX\, Dg\, \exp\Big[-i \int d^2\sigma \sqrt{e^2}\, (e^{-i\theta} e_1{}^a e_1{}^b \partial_a X^i \partial_b X^i - e^{i\theta} e_2{}^a e_2{}^b \partial_a X^i \partial_b X^i$$
$$\underline{-e^{-3i\theta}} e_1{}^a e_1{}^b \partial_a X^D \partial_b X^D + e^{-i\theta} e_2{}^a e_2{}^b \partial_a X^D \partial_b X^D) \Big]$$

Cauchy's integral thm. cannot be applied.

Introduction

<u>*3 different types of world-sheet formulation*</u>

Nambu-Goto type

$$S_{\mathrm{NG}} = -\int d^2\sigma \sqrt{-h}$$

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu, \quad h = \det h_{ab}$$

Schild type

$$S_{\mathrm{Schild}} = -\frac{1}{2}\int d^2\sigma \left(\frac{-h}{e_g} + e_g \right)$$

Quantum mechanically

Equivalent?

Polyakov type

$$S_{\mathrm{P}} = -\frac{1}{2}\int d^2\sigma \sqrt{-g}\, g^{ab} h_{ab}$$

The Euclidean case was discussed long time ago.

[Polyakov '87; Yoneya '97]

Introduction

Perturbative string theory

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\, D\theta\, Dg\; V_{j_1}(k_1)\cdots V_{j_n}(k_n)\; \exp[-S_{\mathrm{P}}^{(\mathrm{E})}]$$

Questions:

- Is it equivalent to the Minkowskian theory?

  ➡ Yes!

- Is it equivalent to the Nambu-Goto-type formulation?

  ➡ Yes!

$$\exp\left[-i\int d^2\sigma\sqrt{-\det h_{ab}}\right]$$

- For a Minkowskian theory, how do we define the Nambu-Goto-type formulation in a path integral?

  ➡ $i\epsilon$ terms select a branch

- Does it have the same features as standard QFT? Causality?

  ➡ $\det h_{ab} > 0$ does not contribute

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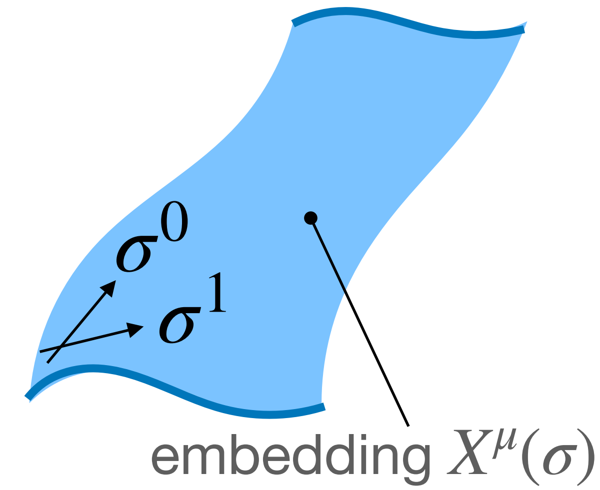
Green-Schwarz formalism



Nambu-Goto-type action

The following respects target-space supersymmetry.

$$S_{\rm GS} = -\frac{1}{2\pi}\int d^2\sigma \Big\{ \underbrace{\sqrt{-h}}_{\text{area of the worldsheet}} - i\varepsilon^{ab}\partial_a X^\mu(\theta^{1T}\Gamma_\mu\partial_b\theta^1 - \theta^{2T}\Gamma_\mu\partial_b\theta^2) + \varepsilon^{ab}\theta^{1T}\Gamma^\mu\partial_a\theta^1\,\theta^{2T}\Gamma_\mu\partial_b\theta^2 \Big\}$$

X^μ: bosons (position of a string) θ^A: Majorana-Weyl fermions $(A = 1,2)$

worldsheet index: $a = 0,1$ target space index: $\mu = 0,\cdots,9$

$$h = \det h_{ab} \quad h_{ab} = \eta_{\mu\nu}\Pi^\mu_a\Pi^\nu_b, \qquad \Pi^\mu_a = \partial_a X^\mu - i(\theta^{1T}\Gamma^\mu\partial_a\theta^1 + \theta^{2T}\Gamma^\mu\partial_a\theta^2)$$

SUSY (10D type II): $\delta^{\rm s}\theta^A = \epsilon^A \qquad \delta^{\rm s}X^\mu = i\epsilon^{AT}\Gamma^\mu\theta^A$

κ symmetry: $\delta^{\rm f}\theta^A = (\mathbf{1} - (-1)^A\tilde\Gamma)\kappa^A(\sigma) \qquad \delta^{\rm f}X^\mu = -i\delta^{\rm f}\theta^{AT}\Gamma^\mu\theta^A$

$$\tilde\Gamma^2 = \mathbf{1}$$

[Green, Schwarz '84]

Hamiltonian and the Polyakov type

The theory has constraints corresponding to gauge symmetry:

$$\left.\begin{aligned} \chi^0_{\mathrm{b}} &= (P_\mu - i(\theta^{1T}\Gamma_\mu\partial_1\theta^1 - \theta^{2T}\Gamma_\mu\partial_1\theta^2))^2 + \Pi^\mu_1\Pi_{1\mu} \approx 0 \\ \chi^1_{\mathrm{b}} &= P_\mu\partial_1 X^\mu + \pi^A\partial_1\theta^A \approx 0 \end{aligned}\right\} \quad \text{(diffeo.)}$$

$$\chi^A_{\mathrm{f}} = \pi^A + i(P_\mu + (-1)^{A+1}\partial_1 X_\mu - (-1)^{A+1} i\theta^{AT}\Gamma_\mu\partial_1\theta^A)(\theta^{AT}\Gamma^\mu) \approx 0 \quad (\kappa \text{ sym.})$$

P_μ: momenta for X^μ $\quad\quad$ π^A: momenta for θ^A $(A = 1,2)$

The Hamiltonian of the system is then

$$\mathscr{H} = \Lambda_0\chi^0_{\mathrm{b}} + \Lambda_1\chi^1_{\mathrm{b}} + \chi^{1\,T}_{\mathrm{f}}\Lambda^1_{\mathrm{f}} + \chi^{2\,T}_{\mathrm{f}}\Lambda^2_{\mathrm{f}}$$

By Legendre-transforming it back, we get the Polyakov-type action

$$S_{\mathrm{P}} = -\frac{1}{2\pi}\int d^2\sigma\left\{\underline{\frac{1}{2}\sqrt{-g}\,g^{ab}h_{ab}} - i\varepsilon^{ab}\partial_a X^\mu(\theta^{1T}\Gamma_\mu\partial_b\theta^1 - \theta^{2T}\Gamma_\mu\partial_b\theta^2) + \varepsilon^{ab}\theta^{1T}\Gamma^\mu\partial_a\theta^1\,\theta^{2T}\Gamma_\mu\partial_b\theta^2\right\}$$

$$\sqrt{-g}\,g^{ab} = \begin{pmatrix} -\dfrac{1}{\Lambda_0} & \dfrac{\Lambda_1}{\Lambda_0} \\ \dfrac{\Lambda_1}{\Lambda_0} & \dfrac{-\Lambda_1^2 + \Lambda_0^2}{\Lambda_0} \end{pmatrix}$$

Schild-type action

$$S_{\mathrm{P}} = -\frac{1}{2\pi}\int d^2\sigma\Big\{\frac{1}{2}\underline{\sqrt{-g}\,g^{ab}h_{ab}} - i\varepsilon^{ab}\partial_a X^\mu(\theta^{1T}\Gamma_\mu\partial_b\theta^1 - \theta^{2T}\Gamma_\mu\partial_b\theta^2)$$
$$+\varepsilon^{ab}\theta^{1T}\Gamma^\mu\partial_a\theta^1\,\theta^{2T}\Gamma_\mu\partial_b\theta^2\Big\}$$

$$= -\frac{h_{11}}{\Lambda_0}\left(\Lambda_1 - \frac{h_{01}}{h_{11}}\right)^2 - \frac{h_{00}h_{11} - h_{01}^2}{\Lambda_0 h_{11}} + \Lambda_0 h_{11}$$

Integrating out Λ_1, we arrive at the Schild-type action

$$S_{\mathrm{Schild}} = -\frac{1}{2\pi}\int d^2\sigma\Big\{\frac{1}{2}\left(\frac{-h}{e_g} + e_g\right) - i\varepsilon^{ab}\partial_a X^\mu(\theta^{1T}\Gamma_\mu\partial_b\theta^1 - \theta^{2T}\Gamma_\mu\partial_b\theta^2)$$
$$+\varepsilon^{ab}\theta^{1T}\Gamma^\mu\partial_a\theta^1\,\theta^{2T}\Gamma_\mu\partial_b\theta^2\Big\}$$

$$e_g = \Lambda_0 h_{11}$$

※ Integrating out e_g brings this back to the Nambu-Goto-type action.

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Overview of the equivalences

$$A_{j_1,\cdots,j_n}(k_1,\cdots,k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\, D\theta\, Dg\, V_{j_1}(k_1)\cdots V_{j_n}(k_n)\, \exp[-S_{\rm P}^{(\rm E)}]$$

Polyakov's Euclidean path int.

$\Updownarrow$ ※ equiv. at least after gauge-fixing κ sym.

Minkowskian path int. w/ the Polyakov-type action

$\Updownarrow$

Schild-type action

$\Updownarrow$

Nambu-Goto-type action

The equivalences hold for critical type IIB and IIA string and critical bosonic string theory on the flat target space.

Path integral—Euclidean to Minkowskian

Let's start with Polyakov's Euclidean path integral
in the case of critical bosonic string theory for simplicity.

$$Z = \int DX\, Dg\, \exp\left[-\frac{1}{2}\int d^2\sigma \sqrt{g}\, g^{ab} h_{ab}\right]$$

Polyakov-type

$$Dg = D\phi \prod_\sigma \frac{2e^\phi\, d\Lambda_1 d\Lambda_2}{(\Lambda_2)^2} \qquad g^{ab} = e^{-\phi}\begin{pmatrix} \frac{\Lambda_1^2+\Lambda_2^2}{\Lambda_2} & -\frac{\Lambda_1}{\Lambda_2} \\ -\frac{\Lambda_1}{\Lambda_2} & \frac{1}{\Lambda_2} \end{pmatrix}$$

$$\| \delta g \|^2 = \frac{1}{2}\int d^2\sigma \sqrt{g}\, g^{ab}\delta g_{bc} g^{cd} \delta g_{da} = \int d^2\sigma\, e^\phi \left(\frac{\delta\Lambda_1^2 + \delta\Lambda_2^2}{(\Lambda_2)^2} + \delta\phi^2 \right)$$

$$= \int DX\, Dg\, \exp\left[-\frac{1}{2}\int d^2\sigma \left\{ \underline{\frac{h_{11}}{\Lambda_2}\left(\Lambda_1 - \frac{h_{12}}{h_{11}}\right)^2} + \frac{h_{11}h_{22}-h_{12}^2}{\Lambda_2 h_{11}} + \Lambda_2 h_{11} \right\}\right]$$

$$= \int DX \left[\prod_\sigma \int_0^\infty \frac{2d\Lambda_2}{\sqrt{\Lambda_2^3 h_{11}}}\right] \exp\left[-\frac{1}{2}\int d^2\sigma \left\{ \frac{h_{11}h_{22}-h_{12}^2}{\Lambda_2 h_{11}} + \Lambda_2 h_{11} \right\}\right]$$

Schild-type

Path integral—Euclidean to Minkowskian

Cauchy's integral thm. equates the path integral to its Minkowskian version by the following deformation of the contour:

$$X^D = e^{i\theta} X^0, \qquad \Lambda_2 h_{11} =: e_g^{(\mathrm{E})} = e^{i\theta} e_g, \qquad \sigma^2 \to \sigma^0 \qquad \left(\theta : 0 \to \frac{\pi}{2} \quad \text{for } e_g > 0\right)$$

$$Z = \int DX \left[\prod_\sigma \int_0^\infty \frac{2de_g^{(\mathrm{E})}}{e_g^{(\mathrm{E})\,3/2}} \right] \exp\left[\underline{-\frac{1}{2} \int d^2\sigma \left\{ \frac{\boxed{h_{11}h_{22} - h_{12}^2}}{e_g^{(\mathrm{E})}} + e_g^{(\mathrm{E})} \right\}} \right]$$

$$h = \frac{1}{2}\{(\varepsilon^{ab}\partial_a X^i \partial_b X^j)^2 + 2(\varepsilon^{ab}\partial_a X^D \partial_b X^i)^2\} > 0$$

$$-\frac{1}{2} \int d^2\sigma \left\{ \frac{e^{-i\theta}(\varepsilon^{ab}\partial_a X^i \partial_b X^j)^2 + 2e^{i\theta}(\varepsilon^{ab}\partial_a X^0 \partial_b X^i)^2}{2e_g} + e^{i\theta} e_g \right\}$$

Path integral—Euclidean to Minkowskian

Cauchy's integral thm. equates the path integral to its Minkowskian version by the following deformation of the contour:

$$X^D = e^{i\theta}X^0, \qquad \Lambda_2 h_{11} =: e_g^{(\mathrm{E})} = e^{i\theta}e_g, \qquad \sigma^2 \to \sigma^0 \qquad \left(\theta : 0 \to \frac{\pi}{2} \quad \text{for } e_g > 0\right)$$

$$Z = \int DX \left[\prod_\sigma \int_0^\infty \frac{2de_g^{(\mathrm{E})}}{e_g^{(\mathrm{E})\,3/2}}\right] \exp\left[\underline{-\frac{1}{2}\int d^2\sigma \left\{\frac{\boxed{h_{11}h_{22}-h_{12}^2}}{e_g^{(\mathrm{E})}} + e_g^{(\mathrm{E})}\right\}}\right]$$

$$h = \frac{1}{2}\{(\epsilon^{ab}\partial_a X^i \partial_b X^j)^2 + 2(\epsilon^{ab}\partial_a X^D \partial_b X^i)^2\} > 0$$

$$\overset{\theta=\pi/2}{=} -\frac{i}{2}\int d^2\sigma \left\{\frac{-(\epsilon^{ab}\partial_a X^i \partial_b X^j)^2 + 2(\epsilon^{ab}\partial_a X^0 \partial_b X^i)^2}{2e_g} + e_g\right\}$$

$$= \int DX \left[\prod_\sigma \int_0^\infty \frac{-2de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \left\{\frac{-(h_{11}h_{00}-h_{10}^2)}{e_g} + e_g\right\}\right]$$

$$= \int DX \left[\prod_\sigma \int_{-\infty}^\infty \frac{-de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \left\{\frac{-(h_{11}h_{00}-h_{10}^2)}{e_g} + e_g\right\}\right]$$

(※ Another deformation, $\theta : 0 \to -\frac{\pi}{2}$, gives integration over $e_g \in (-\infty,0)$.)

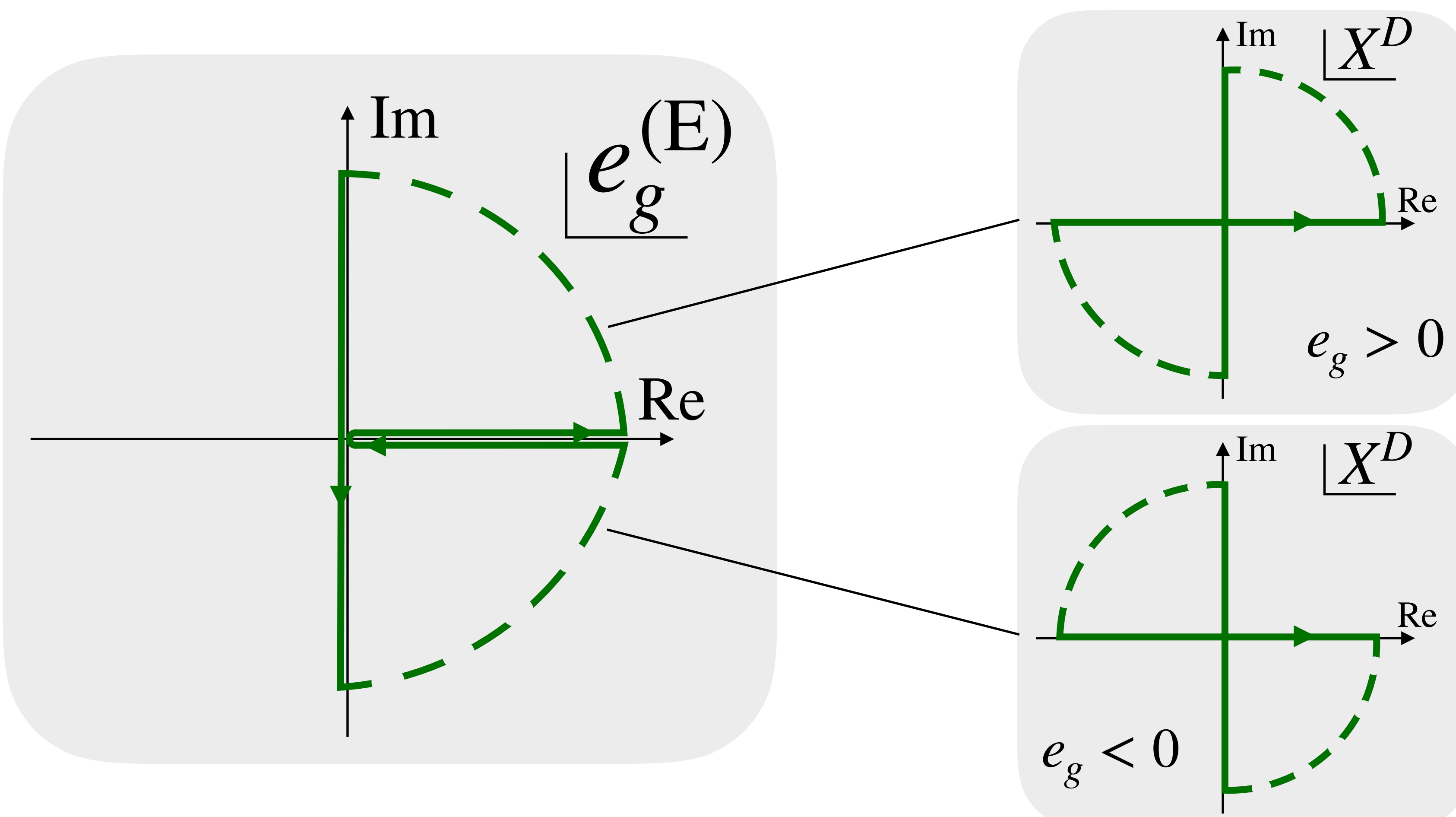
Path integral—Euclidean to Minkowskian

The deformation of the contour:

$$X^D = e^{i\theta} X^0, \qquad \Lambda_2 h_{11} =: e_g^{(\mathrm{E})} = e^{i\theta} e_g, \qquad \sigma^2 \to \sigma^0$$

$$Z = \int DX \left[\prod_\sigma \int_0^\infty \frac{2de_g^{(\mathrm{E})}}{e_g^{(\mathrm{E})\,3/2}} \right] \exp\left[-\frac{1}{2} \int d^2\sigma \left\{ \frac{h_{11}h_{22} - h_{12}^2}{e_g^{(\mathrm{E})}} + e_g^{(\mathrm{E})} \right\} \right]$$

$$= \int DX \left[\prod_\sigma \int_{-\infty}^\infty \frac{-de_g}{(ie_g)^{3/2}} \right] \exp\left[-\frac{i}{2} \int d^2\sigma \left\{ \frac{-(h_{11}h_{00} - h_{10}^2)}{e_g} + e_g \right\} \right]$$



Path integral—Euclidean to Minkowskian

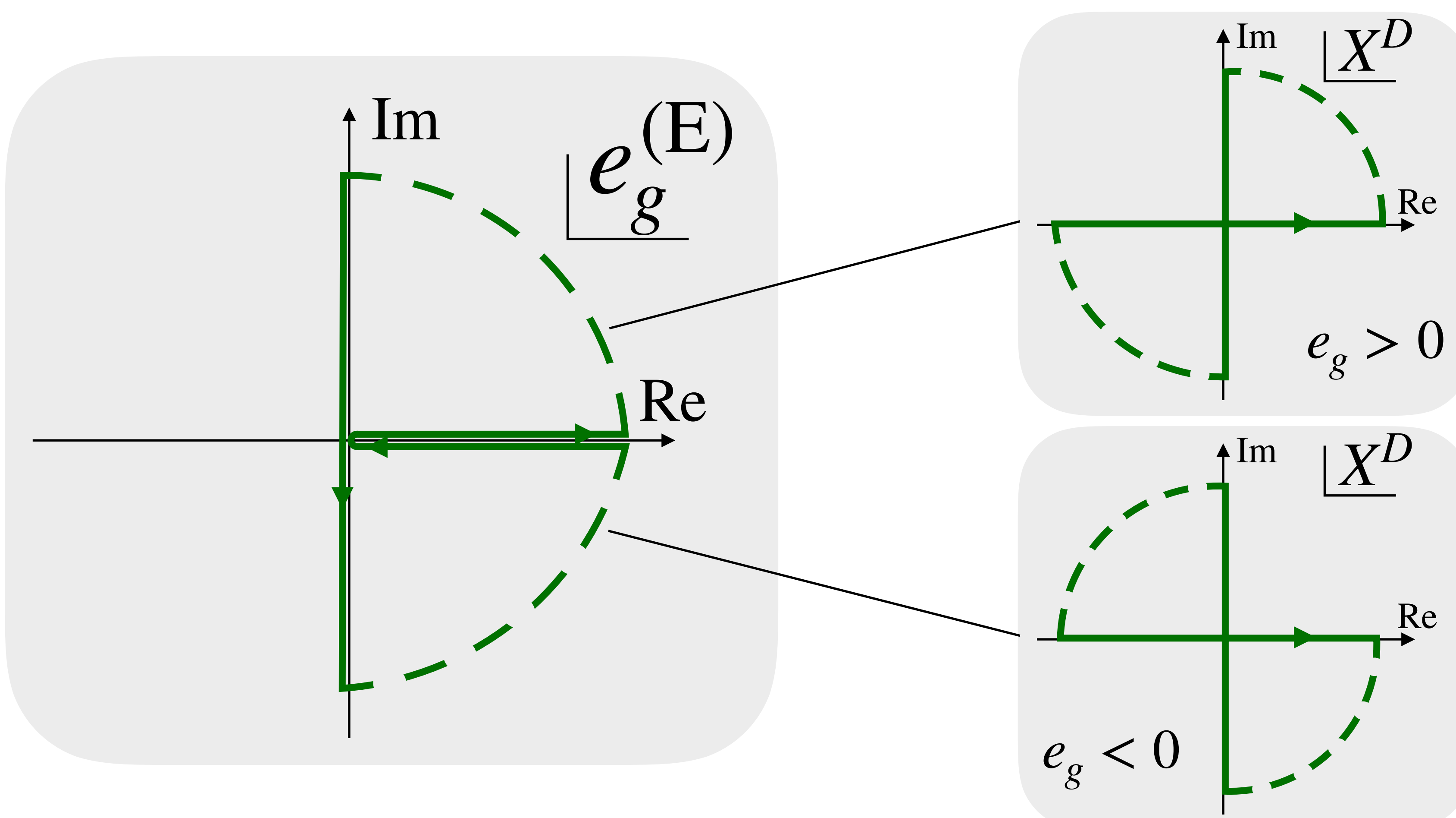
The deformation of the contour:

$$X^D = e^{i\theta} X^0, \qquad \Lambda_2 h_{11} =: e_g^{(\mathrm{E})} = e^{i\theta} e_g, \qquad \sigma^2 \to \sigma^0$$

$$Z = \int DX \left[\prod_\sigma \int_{\mathscr{C}} \frac{-i d e_g^{(\mathrm{E})}}{(-e_g^{(\mathrm{E})})^{3/2}} \right] \exp \left[-\frac{1}{2} \int d^2\sigma \left\{ \frac{h_{11}h_{22} - h_{12}^2}{e_g^{(\mathrm{E})}} + e_g^{(\mathrm{E})} \right\} \right]$$

$$\mathscr{C} : +\infty - i0 \to -0 \to +\infty + i0$$

$$= \int DX \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{-d e_g}{(i e_g)^{3/2}} \right] \exp \left[-\frac{i}{2} \int d^2\sigma \left\{ \frac{-(h_{11}h_{00} - h_{10}^2)}{e_g} + e_g \right\} \right]$$



Path integral — Euclidean to Minkowskian

$$Z = \int DX \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{-de_g}{(ie_g)^{3/2}}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \left\{\frac{-(h_{11}h_{00}-h_{10}^2)}{e_g} + e_g\right\}\right]$$

$$1 = \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{d\Lambda_1}{(i\Lambda_0/h_{11})^{1/2}}\right] \exp\left[\frac{i}{2}\int d^2\sigma \frac{h_{11}}{\Lambda_0}\left(\Lambda_1 - \frac{h_{10}}{h_{11}}\right)^2\right]$$

$$e_g = \Lambda_0 h_{11}$$

$$g^{ab} = e^{-\phi}\begin{pmatrix} -\frac{1}{\Lambda_0} & \frac{\Lambda_1}{\Lambda_0} \\ \frac{\Lambda_1}{\Lambda_0} & \frac{-\Lambda_1^2+\Lambda_0^2}{\Lambda_0} \end{pmatrix}$$

$$= \int DX \left[\prod_\sigma \frac{d\Lambda_0 d\Lambda_1}{(\Lambda_0)^2}\right] \exp\left[-\frac{i}{2}\int d^2\sigma \sqrt{-g}\, g^{ab} h_{ab}\right]$$

Polyakov-type

Polyakov's Euclidean path int. is equivalent to its Minkowskian ver.

[Y.A. '24]

Path integral—Polyakov to Nambu-Goto

The obtained Minkowskian Schild-type path integral effectively contains $i\epsilon$ terms:

$$Z = \int DX \left[\prod_\sigma \int_{-\infty}^{\infty} \frac{-de_g}{(ie_g)^{3/2}} \right] \exp \left[-\frac{i}{2} \int d^2\sigma \left\{ \frac{-h}{e_g} + e_g - i\epsilon |e_g| - i \frac{\tilde{\epsilon}}{|e_g|} \right\} \right]$$

Schild-type

$e_g = \Lambda_0 h_{11}$

Regulators for the convergence of the path integral:

- $i\epsilon$ terms are regarded as terms from the ground state wave function
- gauge invariant
- Λ_a correspond to constraints: $\delta(\chi) = \int_{-\infty}^{\infty} \frac{d\Lambda}{2\pi} e^{i\Lambda\chi - \epsilon|\Lambda|}$ $\rightarrow \Lambda_a \in (-\infty, \infty)$

Path integral—Polyakov to Nambu-Goto

Schild-type

$$\left[\prod_{\sigma}\int_{-\infty}^{\infty}\frac{-de_g}{(ie_g)^{3/2}}\right]\exp\left[-\frac{i}{2}\int d^2\sigma\left\{\frac{-h}{e_g}+e_g-i\epsilon|e_g|-i\frac{\tilde{\epsilon}}{|e_g|}\right\}\right]$$

$$=\prod_{\sigma}\left(\underbrace{\frac{\sqrt{2\pi}}{\sqrt{-h-i\epsilon}}e^{-i\Delta\Sigma\sqrt{-h-i\epsilon}}+\frac{\sqrt{2\pi}}{\sqrt{-h+i\epsilon}}e^{i\Delta\Sigma\sqrt{-h+i\epsilon}}}_{\text{cancel if } h>0}\right)$$

$$=\left[\prod_{\sigma}\sum_{s(\sigma)=\pm1}\frac{\sqrt{2\pi}}{\sqrt{-h-i\epsilon s}}\right]\exp\left[-i\int d^2\sigma\, s\sqrt{-h-i\epsilon s}\right]\qquad\begin{cases}s=1:\text{ F1}\\ s=-1:\text{ anti-F1}\end{cases}$$

Nambu-Goto-type

The Polyakov, Schild and Nambu-Goto types are quantum mechanically equivalent.

The causality is realised by an anti-F1. [Y.A. '24]

“Negative-energy” anti-F-string

What is this anti-F-string? Looks traveling backward in time.

- This is not very weird because there’d be no worldsheet time direction in the first place.

  Time evolution: $\exp\Big[-i\int(\Lambda_0\chi_{\mathrm{b}}^0+\Lambda_1\chi_{\mathrm{b}}^1+\chi_{\mathrm{f}}^{AT}\Lambda_{\mathrm{f}}^A)\,d^2\sigma\Big]\approx 1$

- The anti-string w/ $\Lambda_0<0$ is dual to a string w/ $\Lambda_0>0$ by $\sigma^0\leftrightarrow\sigma^1$.

  $\cdots$ open-closed string duality

  i.e. if $\underbrace{g_{00}}_{\sim -\frac{1}{\Lambda_0}}$ is positive (“wrong” sign), the exchange of σ^a leads us to the “right” sign.

The stringy causality may suggest its interpretation as
an anti-string w/ “negative energy” (the wrong sign),
which corresponds to a “positive-energy” string, like standard QFT.

Propagator for space-like separation

In standard QFT, a space-like propagator is non-zero, which seems contradicting the stringy causality?

In string theory, a 2-point function is

$$V(k,\sigma) \sim \int d^2\sigma\, e^{ik\cdot X(\sigma)}$$

on-shell

$$A_2(\boldsymbol{k},\boldsymbol{k}') = \int d\mu\ \exp[iS_{\rm NG}]\, V(k')\, V(k) \ \propto\ 2k^0\, \delta^{D-1}(\boldsymbol{k}-\boldsymbol{k}')$$

since the Minkowskian theory is equivalent to the Euclidean ver.

Its (Lorentz invariant) Fourier transform gives a propagator:

$$\tilde{A}_2(x,x') = \int \frac{d^{D-1}\boldsymbol{k}}{2k^0} \int \frac{d^{D-1}\boldsymbol{k}'}{2k'^0} e^{i(k\cdot x - k'\cdot x')} A_2(\boldsymbol{k},\boldsymbol{k}') \propto \int \frac{d^{D-1}\boldsymbol{k}}{2k^0} e^{ik\cdot(x-x')}$$

$\cdots$ No contradiction

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Toward the non-pert. definition

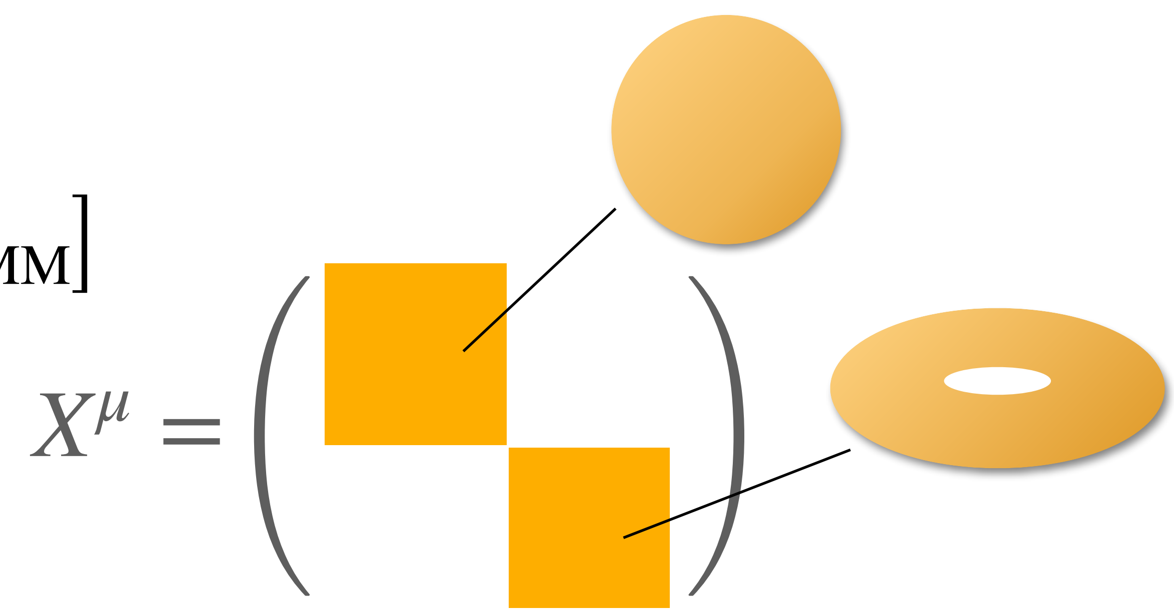
The path integral of perturbative string theory:

$$A(k_1, \cdots, k_n) = \sum_{\chi=2,0,-2,\cdots} g_s^{-\chi} \int DX\, D\theta\, De_g\, V(k_1)\cdots V(k_n)\, \exp\left[iS_{\mathrm{Schild}}\right]$$

… This is merely perturbation theory around the 10D flat spacetime.

Matrix regularisation ⬇

$$A(k_1, \cdots, k_n) = \int d\mu\, V(k_1)\cdots V(k_n)\, \exp\left[iS_{\mathrm{MM}}\right]$$



We expect the matrices describe multi-body systems of superstrings.

Matrix regularisation

A map of functions on a compact space to matrices [Hoppe '82]

$$f(\sigma) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} \underline{Y_{lm}(\sigma)} \longmapsto \sum_{l=0}^{N-1} \sum_{m=-l}^{l} f_{lm} (Y_{lm})_{ij} = f_{ij}$$

fn. on S^2 matrix

spherical harmonics

Matrix regularisation of the Schild-type theory

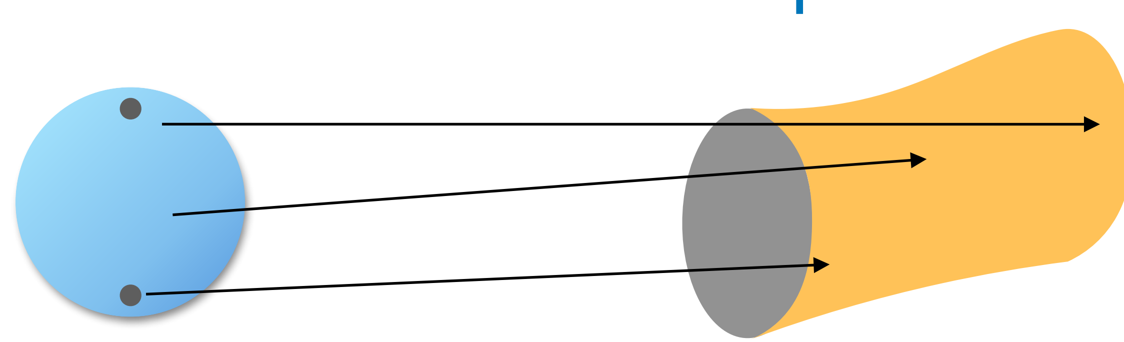
1. Matrix Regularisation after the Wick rotation

   Cauchy's thm. equates the Euclidean theory to the original.

   The regularisation is manifestly well-defined because the worldsheet and the target space are Riemannian.

2. Matrix Regularisation w/o Wick rotation

   Though the target space is Lorentzian, the worldsheet coordinates are just parameters. Consider compact worldsheet with <u>punctures</u>.



Toward the non-pert. definition

We fix the fermionic gauge of the Schild-type theory by

$$\varphi = \theta^1 + i\theta^2 = 0 \qquad \psi = \theta^1 - i\theta^2$$

Then we obtain

$$\{f, g\}_{\hat{\mathrm{P}}} := \varepsilon^{ab}\partial_a f \partial_b g$$

$$S_{\mathrm{Schild}} = \frac{1}{2\pi}\int d^2\sigma \left[\frac{1}{4e_g}\{X^\mu, X^\nu\}^2_{\hat{\mathrm{P}}} + 2i\psi^T\Gamma_\mu\{X^\mu, \psi\}_{\hat{\mathrm{P}}} - \frac{e_g}{2}\right]$$

[Ishibashi, Kawai, Kitazawa, Tsuchiya '96]

By matrix regularisation, $\{\,\cdot\,,\cdot\,\}_{\hat{\mathrm{P}}} \mapsto \frac{N}{i}[\,\cdot\,,\cdot\,]$, $\quad \frac{1}{\pi}\int d^2\sigma \mapsto \frac{1}{N}\mathrm{tr}$,

with $e_g \mapsto -Y$, without Wick rotation

$$\int DX\, D\psi\, De_g\, e^{iS_{\mathrm{Schild}}} \longmapsto \int DX\, D\psi\, DY\, e^{iS_{\mathrm{NBI}}}$$

$$S_{\mathrm{NBI}} = N\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}[X^\mu, X^\nu]^2 + \frac{1}{2}\psi^T\Gamma_\mu[X^\mu, \psi] + Y + \frac{i}{N}(N+\tfrac{1}{2})\ln(-iY)\right)$$

cf. [Fayyazuddin, Makeenko, Olesen, Smith, Zarembo '96]

※ The Euclidean IKKT is obtained by MR after Wick rot. w/ $e_g^2 = 1$

Minkowskian "dielectric" NBI IKKT model

$$S_{\mathrm{NBI}} = N\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}[X^\mu, X^\nu]^2 + \frac{1}{2}\psi^T\Gamma_\mu[X^\mu, \psi] + Y + \frac{i}{N}(N+\tfrac{1}{2})\ln(-iY)\right)$$

cf. [Fayyazuddin, Makeenko, Olesen, Smith, Zarembo '96]

Unlike the IKKT model,
this NBI-type IKKT model explicitly holds the "causality" property:

$$\int DY \exp\left[iN\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}M + \frac{1}{N^2}Y + \frac{i}{N}\left(N+\frac{1}{2}\right)\ln(-iY) + i\epsilon Y^2 + i\tilde{\epsilon}Y^{-2}\right)\right]$$

$$\propto \Delta(m)^{-1} \det_{i,j}\left[\left(iN\frac{\partial}{\partial\alpha}\right)^{j-1}\left(\frac{e^{-i\sqrt{m_i - i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i - i\epsilon'}} + \frac{e^{i\sqrt{m_i + i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i + i\epsilon'}}\right)\right]_{\alpha\to 1}$$

$M := [X^\mu, X^\nu]^2$, m_i: the ith eigenvalue of M

This is zero if at least one eigenvalue of M is negative.

⋯ similar to the cancellation in perturbative string theory

Minkowskian “dielectric” NBI IKKT model

The large-N limit reproduces the perturbative string theory up to a measure factor.

$$\int DY \exp\left[iN\,\mathrm{tr}\left(\frac{1}{4}Y^{-1}M+\frac{1}{N^2}Y+\frac{i}{N}\left(N+\frac{1}{2}\right)\ln(-iY)+i\epsilon Y^2+i\tilde{\epsilon}Y^{-2}\right)\right]$$

$$\propto \Delta(m)^{-1}\det_{i,j}\left[\left(iN\frac{\partial}{\partial\alpha}\right)^{j-1}\left(\frac{e^{-i\sqrt{m_i-i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i-i\epsilon'}}+\frac{e^{i\sqrt{m_i+i\epsilon'}\sqrt{\alpha}}}{\sqrt{m_i+i\epsilon'}}\right)\right]_{\alpha\to 1}$$

$$=\left(\frac{N}{2}\right)^N\left[\prod_{i=1}^{N}\sum_{s_i=\pm1}\right]\frac{1}{\prod_{i,j<i}(s_i\sqrt{m_i-i\epsilon' s_i}+s_j\sqrt{m_j-i\epsilon' s_j})}\prod_{i=1}^{N}\frac{e^{-is_i\sqrt{m_i-i\epsilon' s_i}}}{\sqrt{m_i-i\epsilon' s_i}}$$

$$\to\left[\prod_{\sigma}\sum_{s(\sigma)=\pm1}\frac{1}{\sqrt{-h-i\epsilon s}}\right]\mathscr{M}[-h(\sigma),s(\sigma)]\exp\left[-i\int d^2\sigma\, s\sqrt{-h-i\epsilon s}\right]$$

cf. [Fayyazuddin, Makeenko, Olesen, Smith, Zarembo ’96]

$$※\sum_i\sqrt{m_i}\to\int d^2\sigma\sqrt{-\frac{1}{2}\{X^\mu,X^\nu\}^2}=\int d^2\sigma\sqrt{-h}\quad \text{as } N\to\infty$$

⋯ the inverse of matrix regularisation

Euclidean “dielectric” NBI IKKT model

Some remarks on the Euclidean NBI model

$$Z_{\rm NBI}^{(E)} = \int DX\, D\psi\, DY\, \exp\left[-N\,{\rm tr}\left(-\frac{1}{4}Y^{-1}[X^m, X^n]^2 + \frac{1}{2}\psi^T\Gamma_\mu[X^\mu, \psi] + \frac{1}{N^2}Y + \gamma\ln(Y)\right)\right]$$

- It has terms reminiscent of the Penner model.

$$Z_{\rm Penner} = \int DY\, \exp[-Nt\,{\rm tr}[-Y + \ln(Y)]]$$

The NBI model has a critical behaviour at $\gamma \to 1$ for large N.

[Chekhov, Zarembo ’97; Kristjansen, Olesen ’97]

→ The genus expansion of a string amplitude is reproduced?

[Distler, Vafa ’91] [Kristjansen, Olesen ’97]

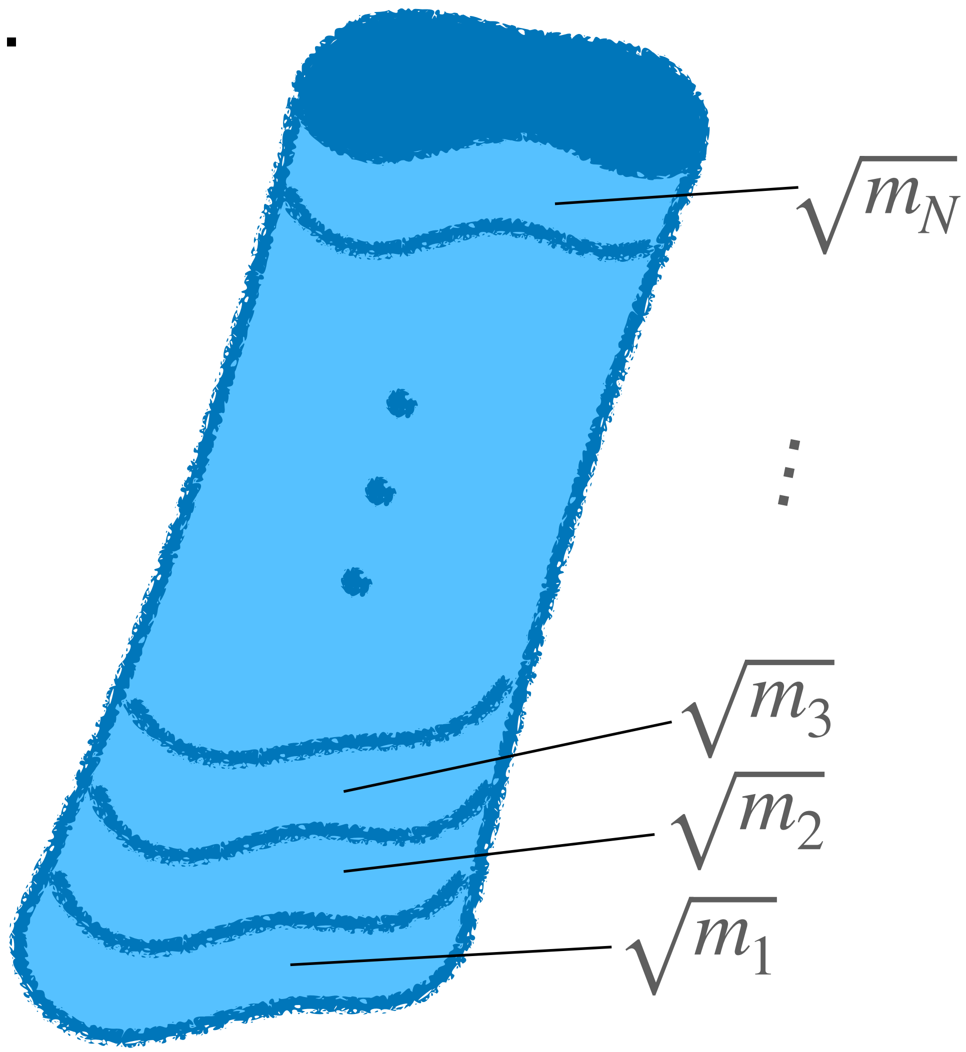
- The Y-part is equivalent to the Kontsevich model. [Ambjorn, Chekhov,’98]

$$Z_{\rm Kontsevich} = \int DY'\, \exp\left[-\gamma'\,{\rm tr}\left[\frac{1}{2}\Lambda Y'^2 - \frac{i}{6}Y'^3\right]\right]$$

$$\Lambda \sim \sqrt{-[X^m, X^n]^2}$$

Minkowskian "dielectric" NBI IKKT model

In summary, the Minkowskian NBI model can be interpreted as a causal matrix model.



The matrix $M^{\frac{1}{2}} = \sqrt{[X^\mu, X^\nu]^2}$ $([X^\mu, X^\nu]^2 > 0)$ approaches a time-like area $\sqrt{-h}$ $(h < 0)$ as $N \to \infty$, and only a time-like area contributes.

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Summary

- The Minkowskian perturbative superstring theory is quantum mechanically equivalent to its Euclidean version in terms of path integration.
- The Polyakov, Schild and Nambu-Goto-type formulations are quantum mechanically equivalent in the case of critical string theory (bosonic & type II).
- Full integration over the worldsheet metric provides the stringy causality. Since configs. with $\det h_{ab} > 0$ don't contribute to the path integral, string propagation with a space-like area is prohibited.
- We obtained the Minkowskian NBI-type IKKT model as a causal matrix model by matrix regularisation of IIB string. This partially answers how we define the IKKT model in the path-integral formalism, but it doesn't uniquely determine the matrix model (even whether it's Euclidean or Minkowskian).

Future work

- It's important to establish the exact relationship between perturbative superstring d.o.f. and matrix d.o.f. cf. SFT: [Fukuma, Kawai, Kitazawa, Tsuchiya '97]
  string states: [Iso, Kawai, Kitazawa '00; Steinacker '16]

  In the matrix model, we have a vertex op. $V^{\Phi} = \mathrm{tr}\, e^{ik_\mu X^\mu}$

  This forms a massless multiplet of type-IIB SUGRA
  by acting the supercharge operator Q onto this vertex.
  [Kitazawa '02; Iso, Terachi, Umetsu '04; Kitazawa, Mizoguchi, Saito '07]

  Perturbative String states ↔ Matrix model Operators

  Amplitudes computed by the vertex operators in the NBI model are expected to reproduce the genus expansion via the matrix reg.
  Is it also obtained by a 1/N expansion?

- A mass term may be essential to define the IKKT model.

  The Polarised IKKT model is singular and subtle. [Hartnoll, Liu '24; Komatsu et al. '24]
  [Benelli '02]

  ➡ Double scaling limit as $\Omega \to 0$ (massless lim.)?