ヤング図から見た FKM模型のGWW相転移について

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based on
arXiv:2509.****

Plan of the talk

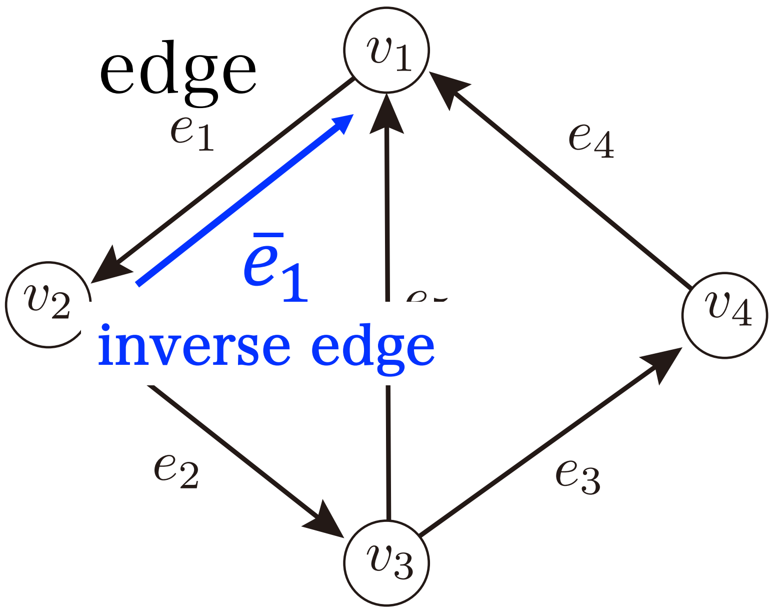
1. グラフゼータ関数概観
2. FKM模型とその性質
3. サイクルグラフ上のFKM模型の厳密解
4. ランダム分割から見たサイクルグラフ上のFKM模型
5. GWW相転移とボーズ・アインシュタイン凝縮

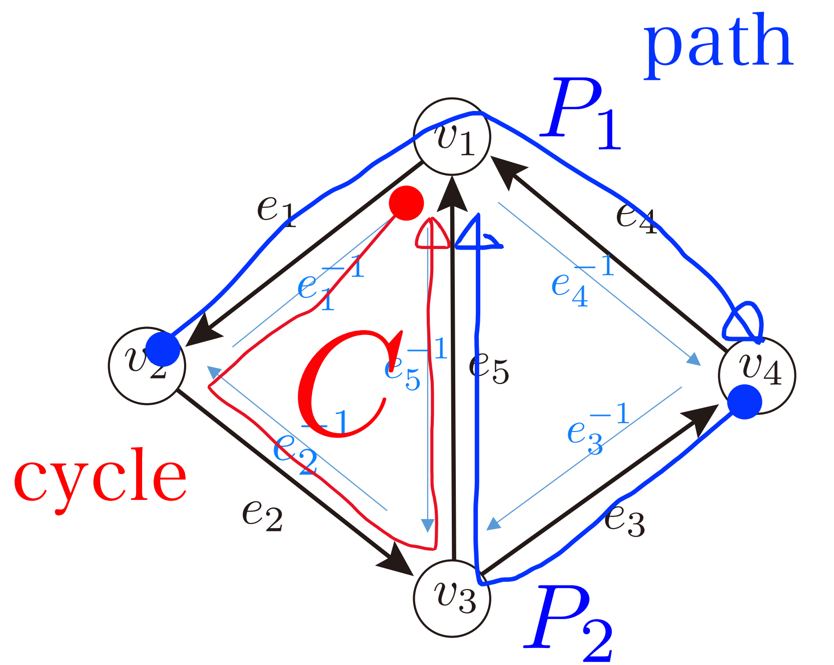
グラフゼータ関数概観

Graph and basic concepts

A graph = An object that vertices are connected by edges

Set of vertices	Set of edges	Source and Target
$V = \{1,2,\cdots,n_V\}$	$E = \{e_1, e_2, \cdots, e_{n_E}\}$	$e = \langle v, v' \rangle \in E \Longrightarrow v = s(e), v' = t(e)$



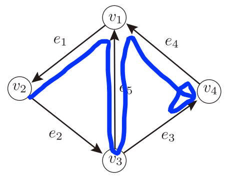


product of paths P_1P_2

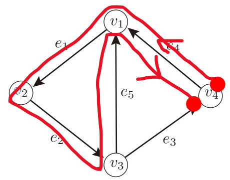
Classification of cycles

Bumps

backtracking



tail



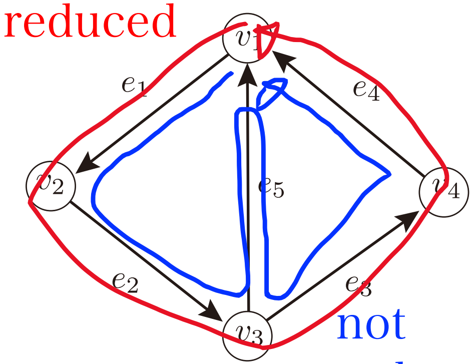
bump

Important concepts on cycles

reduced cycle

cycle without a bump

reduced

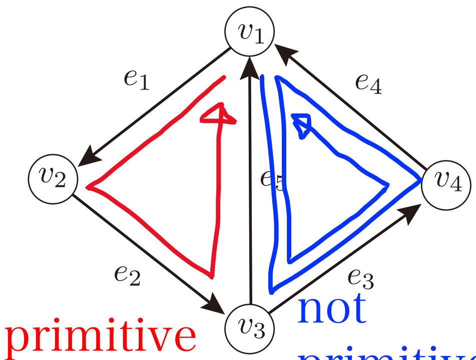


not reduced

primitive cycle

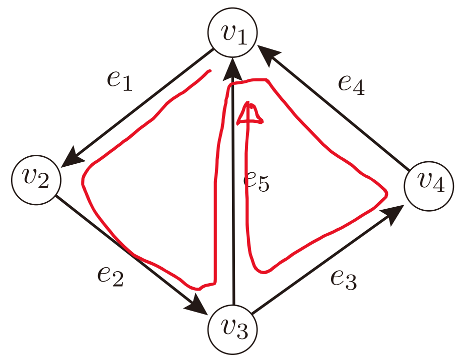
$$C \neq B^n$$

Note: this is also primitive



primitive

not primitive



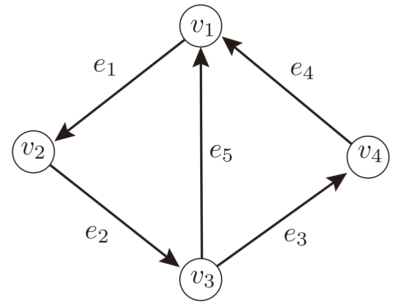
Matrices associated with a graph

Adjacency matrix

$$A_{vv'} = \sum_{\boldsymbol{e}\in E_D} \delta_{\langle v,v'\rangle,\boldsymbol{e}} \,. \quad (v, v' \in V)$$

$$A_{\mathrm{DT}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(example)
Double Triangle (DT)



Degree matrix

$$D \equiv \mathrm{diag}_{v\in V}(\deg(v)) \quad Q \equiv D - 1$$

$$D_{\mathrm{DT}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Edge adjacency matrix

$$W_{\mathbf{e}\mathbf{e}'} \equiv \begin{cases} 1 & \text{if } t(\mathbf{e}) = s(\mathbf{e}') \text{ and } \mathbf{e}'^{-1} \neq \mathbf{e} \\ 0 & \text{others} \end{cases} . \quad (\mathbf{e}, \mathbf{e}' \in E_D)$$

$W_{\mathrm{DT}} =$

	e_1	e_2	e_3	e_4	e_5	e_1^{-1}	e_2^{-1}	e_3^{-1}	e_4^{-1}	e_5^{-1}
e_1	0	1	0	0	0	0	0	0	0	0
e_2	0	0	1	0	1	0	0	0	0	0
e_3	0	0	0	1	0	0	0	0	0	0
e_4	1	0	0	0	0	0	0	0	0	1
e_5	1	0	0	0	0	0	0	0	1	0
e_1^{-1}	0	1	0	0	0	0	0	0	1	1
e_2^{-1}	0	0	0	0	0	1	0	0	0	0
e_3^{-1}	0	0	0	0	1	0	1	0	0	0
e_4^{-1}	0	0	0	0	0	0	0	1	0	0
e_5^{-1}	0	0	1	0	0	0	1	0	0	0

The Ihara zeta function

cf) Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

Ihara zeta function

$$\zeta_G(q) \equiv \prod_{[C]:\text{primitive reduced}} \frac{1}{1-q^{|C|}}$$

length of the cycle

Ihara zeta function is expressed as the reciprocal of a finite polynomial

Vertex expression Ihara 1966

$$\zeta_G(q) = (1-q^2)^{-(n_E - n_V)} \det\left(I - qA + q^2 Q\right)^{-1}$$

Edge expression Hashimoto 1990, Bass 1992

$$\zeta_G(q) = \det(1 - qW)^{-1}$$

Is Ihara zeta function a zeta function?

(1) Euler product?

$$\zeta_G(q) \equiv \prod_{[C]:PR} \frac{1}{1-q^{|C|}}$$

(2) functional equation?

- If the graph is (t+1)-regular, $\zeta_G(q) = (1-q^2)^{-(n_E-n_V)} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)^{-1}$
- Completed Ihara zeta： $\xi_G(q) \equiv (1-q^2)^{n_E-\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}}\zeta_G(q) = (1-q^2)^{\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)$

$$\xi_G\left(\frac{1}{tq}\right) = (-1)^{n_V}\xi_G(q)$$

(3) Riemann's hypothesis?

If the graph is Ramanujan, non-trivial zeros of $\zeta_G(t^{-s})$ are only on $Re(s) = \frac{1}{2}$

(Ramanujan graph：(t+1)-regular and the eigenvalues of A except for $t+1$ satisfies $\lambda^2 < 4t$)

proof

Ihara zeta function of (t+1)-regular graph:

$\zeta_G(q) = (1-q^2)^{n_V-n_E} \det\left((1-tq^2)\mathbf{1}_{n_V} - qA\right) = (1-q^2)^{n_V-n_E}\prod_\lambda (tq^2 - \lambda q + 1)$ ➡ zeros：$q = \frac{\lambda \pm \sqrt{\lambda^2-4t}}{2t} \equiv t^{-s_\pm}$

If $\lambda^2 - 4t < 0$, since $s_- = s_+^*$, $t^{-s_+}\cdot t^{-s_-} = t^{-s_+-s_-} = t^{-2Re(s_+)} = t^{-1}$

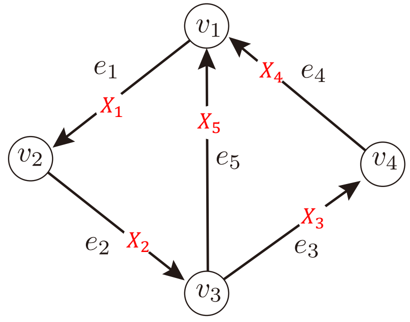
Matrix weighted Ihara zeta function

Ohta-S.M. 2022

cf) Mizuno, Sato 2003,2006

preparation

- regular matrix X_e (size K) on each edge e
- $X_{e^{-1}} = X_e^{-1}$
- $X_C \equiv X_{e_{i_1}} \cdots X_{e_{i_n}}$ for $C = e_{i_1} \cdots e_{i_n}$



Matrix weighted Ihara zeta function Ohta-S.M. 2022

$$\zeta_G(q;X) \equiv \prod_{C \in [\mathcal{P}_R]} \det\left(1_K - q^{|C|} X_C\right)^{-1}$$

matrix weighted adjacency matrices and the matrix weighted Ihara zeta function

$$A(X)_{vv'} = \begin{cases} X_e & \langle v, v' \rangle = e \\ 0 & \text{others} \end{cases} \qquad (W_X)_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} X_{\boldsymbol{e}} & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e}') \text{ and } \boldsymbol{e}'^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}$$

$$\zeta_G(q;X) = \left(1-q^2\right)^{-K(n_E - n_V)} \det\left(1_{Kv_N} - qA_X + q^2 Q\right)^{-1} \quad \text{: vertex expression}$$

$$= \det\left(1_{2Kn_E} - qW_X\right)^{-1} \quad \text{: edge expression}$$

FKM模型とその性質

FKM model

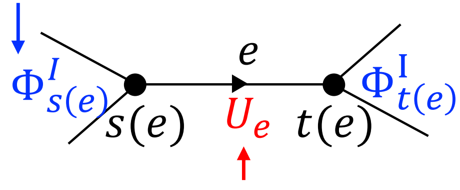
Arefeva 1993
Ohta-S.M. 2023

Fundamental Kazakov-Migdal (FKM) model on a general graph

$$S = \sum_{v\in V} m_v^2 \Phi_v^{\dagger I} \Phi_{vI} - q \sum_{e\in E} \left(\Phi^{\dagger}_{s(e)}{}^{I} U_e \Phi_{t(e)I} + \Phi^{\dagger}_{t(e)}{}^{I} U_e^{\dagger} \Phi_{s(e)I} \right)$$

$$(I = 1, \cdots, N_f)$$

fundamental scalars
$(I = 1, \cdots, N_f)$



unitary matrix
(color N_c)

cf) KM model on the graph

Kazakov-Migdal 1992
Ohta-S.M. 2022

$$S_{\rm KM} = {\rm Tr}\left\{ \frac{m_0^2}{2} \sum_{v\in V} \Phi_v^2 + q \sum_{e\in E} \left(\frac{r}{2}\left(\Phi_{s(e)}^2 + \Phi_{t(e)}^2\right) - \Phi_{s(e)} U_e \Phi_{t(e)} U_e^{\dagger} \right) \right\}$$

The partition function after tunning the mass parameters: $m_v^2 = 1 + (\deg v + 1) q^2$

$$Z_G = \mathcal{N} \int \prod_{e\in E} dU_e \, \zeta_G(q;U)^{N_f}$$

$$\left(\mathcal{N} = (2\pi)^{N_f N_c n_V} \left(1 - q^2\right)^{N_f N_c (n_E - n_V)} \right)$$

FKM model is described by the unitary matrix weighted graph zeta function

Effective action and the relation to the Wilson action

$$\zeta_G(q;U) = \exp\left(\sum_{C\in[\Pi_+]}\sum_{n=1}^{\infty}\frac{q^n}{n}\left(\mathrm{Tr}\,U_C^n + \mathrm{Tr}\,U_C^{\dagger\,n}\right)\right)$$

$$S_{\mathrm{eff}}(U) = -N_f\sum_{C\in[\Pi_+]}\sum_{n=1}^{\infty}\frac{q^n}{n}\left(\mathrm{Tr}\,U_C^n + \mathrm{Tr}\,U_C^{\dagger\,n}\right)$$

$\gamma \equiv N_f/N_c \quad q\to 0, \quad \gamma\to\infty, \quad \lambda\equiv\frac{1}{\gamma q^l}$: fixed $\quad$ (l : minimal length of the cycles)

$$S_{\mathrm{eff}}(U) \to -\frac{N_c}{\lambda}\sum_{C\,:\,\text{minimal length}}\left(\mathrm{Tr}\,U_C + \mathrm{Tr}\,U_C^{\dagger}\right)$$

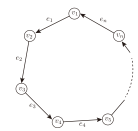
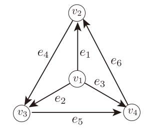
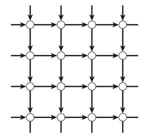
FKM model is a generalization of the usual lattice gauge theory

Duality of the FKM model

Ohta-S.M. 2024

<u>functional equation of the U-Ihara zeta function for a $(t+1)$-regular graph</u>

$$\zeta_G(1/tq;U) = (tq^2)^{n_V N_c} \left(\frac{-tq^2(1-q^2)}{1-t^2q^2} \right)^{(n_E - n_C)N_c} \zeta_G(q;U)$$

The FKM model on a regular graph is **self-dual**

<u>"functional relation" for an irregular graph</u> Ohta-S.M. 2024

$\left(\tilde{Q} \equiv \mathrm{diag}_e(\deg s(e) - 1)\right)$

For an arbitrary ω,

$$\zeta_G(1/\omega q;U) \propto \det\left(1 - \omega q \left(\tilde{Q}^{-1} W_U - (1-\tilde{Q}^{-1}) J_U\right)\right)^{-1} = \exp\left(\sum_{C \in [\Pi_+]} \sum_{n=1}^{\infty} {}^{\exists} f_{C,n}(q) \left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger n} \right) \right)$$

a matrix weighted Bartholdi zeta function with (unfamiliar) weights

The FKM model on an irregular graph has also a **dual expression**.

Result of numerical simulations

Regular graph

(1) Tetrahedron

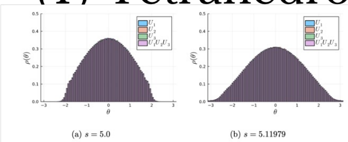
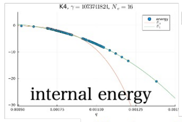
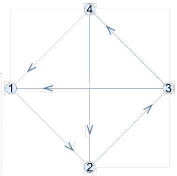
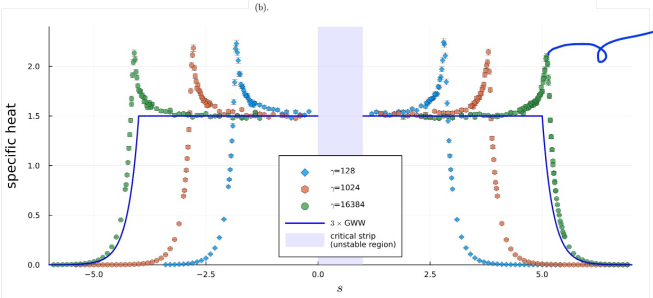


Figure 12: The eigenvalue distribution of U_1, U_2, U_3 and $U_1U_2U_3$ in the FKM model on K_4 at $s = 5.0$, the phase transition point of the GWW model (a), and at $s = 5.11979$, the value of s closest to the peak of the pointed elbow of the specific heat in Fig. 11 (b).







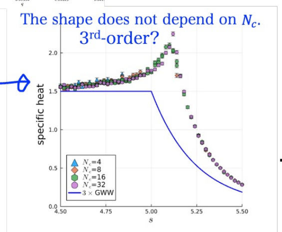


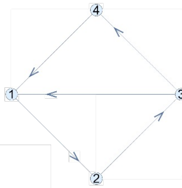
Figure 13: The specific heat of the FKM model on K_4 near the phase transition point with varying $N_c = 4, 8, 16, 32$ and fixing $\gamma = 16384$.

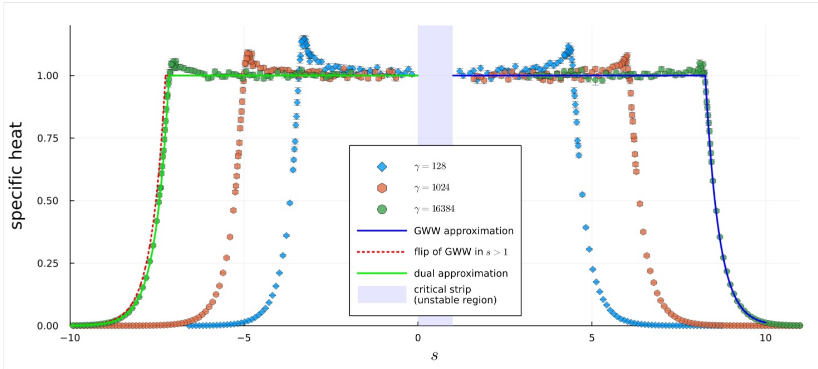
$(q = \omega^{-s})$

functional relation: $s \leftrightarrow -s$

Irregular graph

(2) Double Triangle



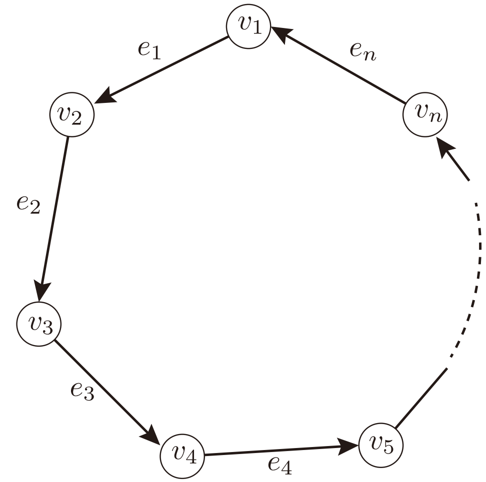


サイクルグラフ上の FKM模型の厳密解とGWW相転移

FKM model on the cycle graph

Ihara zeta function on the cycle graph

$$\zeta_{C_n}(q) = \prod_{[C]} \frac{1}{1-q^{|C|}} = \frac{1}{(1-q^n)^2}$$



After gauge fixing : $U_2 = \cdots = U_n = 1$ $(U_1 \equiv U, \alpha \equiv q^n)$

$$Z_{C_n} = \mathcal{N} \int dU \frac{1}{|\det(1-\alpha U)|^{2N_f}}$$

$$= \mathcal{N} \int dU e^{N_c \sum_{m=1}^{\infty} \frac{\gamma \alpha^m}{m} \mathrm{Tr}\left(U^m + U^{-m}\right)} \qquad (N_f = \gamma N_c)$$

Unitary matrix model

$$S_{\mathrm{eff}} = -N_c \sum_{m=1}^{\infty} \frac{\gamma \alpha^m}{m} \mathrm{Tr}\left(U^m + U^{-m}\right) \xrightarrow[\gamma \to \infty,\ \alpha \to 0,\ \lambda \equiv 1/\gamma\alpha]{} -\frac{N_c}{\lambda} \mathrm{Tr}\left(U + U^{\dagger}\right)$$

GWW model

GWW phase transition on the cycle graph

Effective action of the phase of the eigenvalues

$$S_{\text{eff}}[\theta] = -\log\left|\sin\frac{\theta_j-\theta_k}{2}\right| + N_f\sum_i \log\left(1-2q^n\cos\theta_i+q^{2n}\right)$$

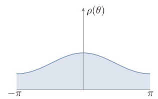
Saddle point equation in $N_c \to \infty$

$$⨍ dy\,\cot\left(\frac{\theta(x)-\theta(y)}{2}\right) = ⨍_{-\theta_0}^{\theta_0} d\theta'\,\rho(\theta')\cot\left(\frac{\theta-\theta'}{2}\right) = \gamma\frac{2\alpha\sin\theta}{1-2\alpha\cos\theta+\alpha^2} \qquad \left(\rho(\theta)\equiv\frac{1}{N_c}\sum_{i=1}^{N_c}\delta(\theta-\theta_i)\right)$$

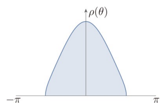
Eigenvalue density in large N_c

cf) Watanabe-san's talk

$$\rho(\theta) = \begin{cases} \dfrac{1}{2\pi}\left(1+2\gamma\dfrac{\alpha\cos\theta-\alpha^2}{1-2\alpha\cos\theta+\alpha^2}\right), & (\theta_0=\pi) \\ \dfrac{2(\gamma-1)\alpha}{\pi}\dfrac{\cos\frac{\theta}{2}}{1-2\alpha\cos\theta+\alpha^2}\sqrt{\sin^2\dfrac{\theta_0}{2}-\sin^2\dfrac{\theta}{2}}, & (\theta_0<\pi) \end{cases}$$



(partially) (de)confined



deconfined

$$\left(\sin^2\frac{\theta_0}{2} = \frac{(1-\alpha)^2}{4\alpha}\frac{2\gamma-1}{(\gamma-1)^2}\right)$$

GWW phase transition on the cycle graph

Free energy

$$F_{C_n} \equiv -\lim_{N_c\to\infty}\frac{1}{N_c^2}\log Z_{C_n} = \begin{cases} F^-_{C_n} \equiv \gamma^2\log\left(1-\alpha^2\right) & (0<\alpha\le\alpha^*) \\ F^+_{C_n} \equiv (2\gamma-1)\log(1-\alpha)+\frac{1}{2}\log\alpha+f(\gamma) & (\alpha^*<\alpha\le 1)\end{cases} \quad \left(\alpha^*=\frac{1}{2\gamma-1}\right)$$

$$\left(f(\gamma)=(\gamma-1)^2\log\left(1-\frac{1}{\gamma}\right)-2\left(\gamma-\frac{1}{2}\right)^2\log\left(1-\frac{1}{2\gamma}\right)\right)$$

Internal energy

$$E_{C_n} \equiv \gamma\frac{\partial F_G}{\partial\gamma} = \begin{cases} 2\gamma^2\log(1-\alpha^2) & (0<\alpha\le\alpha^*) \\ 2\gamma\log(1-\alpha)+e_1(\gamma) & (\alpha^*<\alpha<1)\end{cases}$$

$$\left(e_1(\gamma)\equiv 2\gamma\left(\gamma-1\right)\log\left(1-\frac{1}{\gamma}\right)-2\gamma(2\gamma-1)\log\left(1-\frac{1}{2\gamma}\right)\right)$$

Specific heat

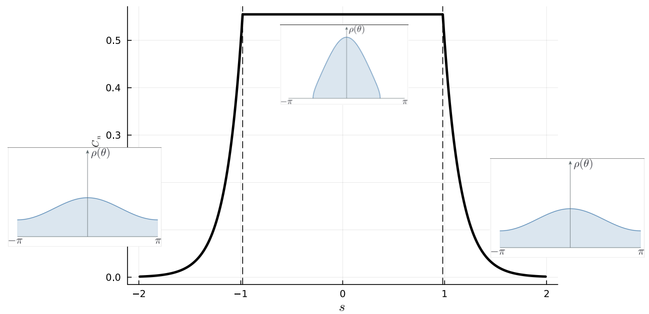
$$C_{C_n} \equiv -\gamma^2\frac{\partial^2 F_G}{\partial\gamma^2} = \begin{cases} -2\gamma^2\log(1-\alpha^2) & (0<\alpha\le\alpha^*) \\ -2\gamma^2\log\dfrac{4\gamma(\gamma-1)}{(2\gamma-1)^2} & (\alpha^*<\alpha<1)\end{cases}$$

Specific heat Including the dual side

3rd order GWW phase transition in both of large/small α

specific heat

$$s=-\log q$$



※ $n=3, \gamma=10 \Rightarrow s_* \simeq \pm 0.98$

ランダム分割から見た
サイクルグラフ上のFKM模型

From unitary matrix to Young diagram

Character of the unitary group

$$\chi_R(U) \equiv \mathrm{Tr}\, U_R$$

Duality of a class function

$f(PUP^{-1}) = f(U)$

$$\sum_{R \in Y_N} \chi_R(U)\chi_R(V) = \delta(UV^\dagger - 1)$$

$$f(U) = \sum_R c(R)\chi_R(U) \qquad c(R) = \int dU\, f(U)\chi_R(U^\dagger)$$

$$\int dU\, \chi_R(U)\chi_{R'}(U^\dagger) = \delta_{RR'}$$

FKM on the cycle graph in terms of Young diagrams

Unitary matrix description

$$Z_{C_n} = \int dU |f(U)|^2 \qquad f(U) = e^{N_f \sum_{m=1}^{\infty} \frac{1}{m} \mathrm{Tr} U^m}$$

Fourier transformation

$$f(U) = e^{N_f \sum_{m=1}^{\infty} \frac{1}{m} \mathrm{Tr} U^m} = \sum_R s_R(x) \chi_R(U) \qquad x \equiv (\overbrace{q^n, \cdots, q^n}^{N_f})$$

Young diagram discription

$$Z_{C_n} = \sum_{R \in Y_{N_c}} |s_R(x)|^2 = \sum_{R \in Y_{N_c}} (\dim_{N_f} R)^2 q^{n|R|}$$

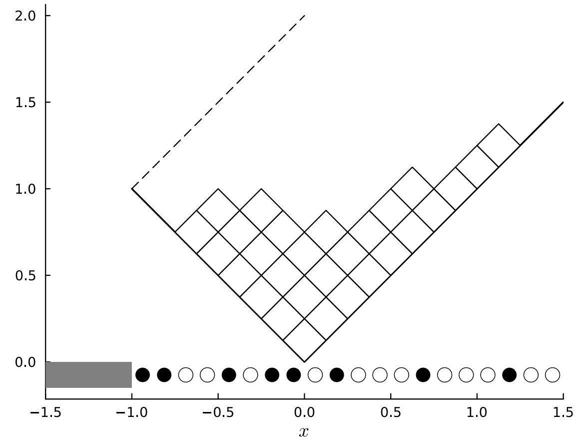
Effective action via Young diagram

Young diagram ``bosonic'' picture

$$R = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N_c} \geq 0\}$$

Maya diagram ``fermionic'' picture

$$n_i \equiv \lambda_i - i \qquad n_1 > n_2 > \cdots > n_{N_c} > -N_c$$



Density of the black dots in the Maya diagram

$$\tilde{\rho}(x) \equiv \frac{1}{N_c}\sum_{i=1}^{N_c} \delta(x - n_i/N_c)$$

Effective action of FKM on C_n in n_i

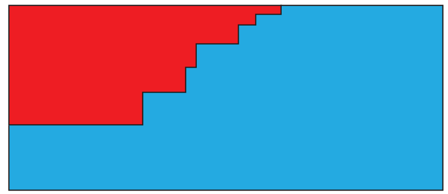
$$\begin{aligned}\tilde{S}[\vec{n}] &= -\sum_{i\neq j} \log|n_i - n_j| - 2\sum_i \log\frac{(n_i+N_f)!}{(n_i+N_c)!} - 2\log\alpha\sum_i n_i \\ &= -N_c^2\left(\int_0^1 dx dx' \,\log|x-x'|\,\tilde{\rho}(x)\tilde{\rho}(x') \right.\\ &\qquad \left. - \int_0^1 dx\,\left(x\log(\gamma^2\alpha^2) + 2\left((\gamma+x)\log(1+\frac{x}{\gamma}) - (1+x)\log(1+x)\right)\right)\tilde{\rho}(x)\right)\end{aligned}$$

Strong/weak duality in Young diagram

$$\tilde{S}[\vec{n}, \alpha] = -\sum_{i \neq j} \log|n_i - n_j| - 2\sum_i \log\frac{(n_i + N_f)!}{(n_i + N_c)!} - \log\alpha^2 \sum_i n_i$$

<u>dual variable of n_i</u>

$$\tilde{n}_i \equiv -N_f - N_c - n_{N_c - i} - 1$$



- $\log|n_i - n_j| = \log|\tilde{n}_{N_c - i} - \tilde{n}_{N_c - j}|$
- $\dfrac{(N_f + n_i)!}{(N_c + n_i)!} = (N_c + n_i + 1)\cdots(N_f + n_i) = (-1)^{N_f - n_c}\dfrac{(N_f + \tilde{n}_{N_c - i})!}{(N_c + \tilde{n}_{N_c - i})!}$
- $-\log\alpha^2 \sum_i n_i = -\log\alpha^{-2} \sum_i \tilde{n}_i + \text{const.}$

$$\tilde{S}[\vec{n}, \alpha] = \tilde{S}[\vec{\tilde{n}}, \alpha^{-1}] \qquad \text{up to const.}$$

Exact solution in large N_c limit

Saddle point equation in $N_c \to \infty$

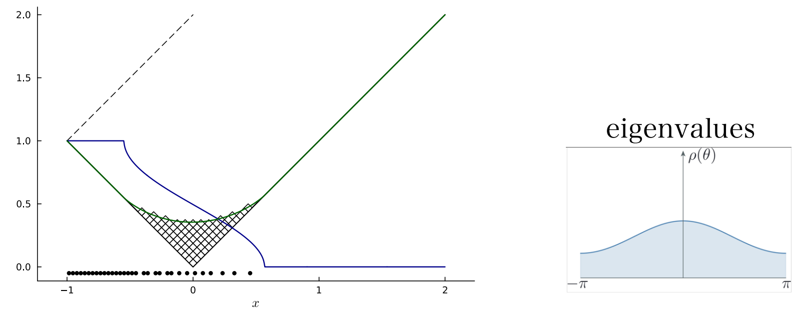
$$\fint dx' \, \frac{\tilde{\rho}(x')}{x'-x} = \log\left(\frac{x+\gamma}{x+1}\,\alpha\right)$$

Solution

$0 < \alpha < \alpha^*$ (confined)

$$\tilde{\rho}(x) = \begin{cases} 1 & (-1 \le x < a) \\ \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{Cs+1}{\sqrt{C^2-1}\sqrt{1-s^2}}\right) & (a \le x \le b) \end{cases}$$

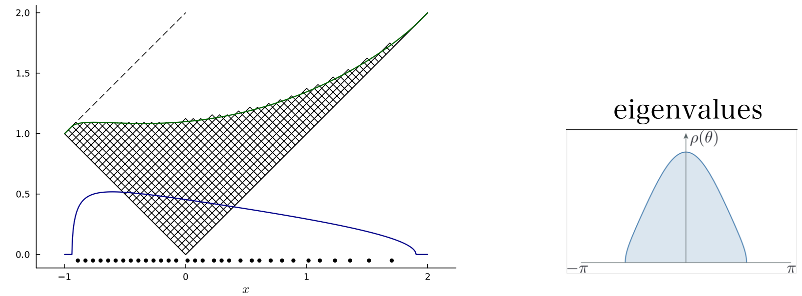
$$a = \frac{-2\gamma\alpha}{1+\alpha}, \quad b = \frac{2\gamma\alpha}{1-\alpha}, \quad C \equiv \frac{\alpha+\alpha^{-1}}{2}$$



$\alpha^* < \alpha < 1$ (deconfined)

$$\tilde{\rho}(x) = \frac{1}{\pi}\arctan\left(\frac{C_1 s+1}{\sqrt{C_1^2-1}\sqrt{1-s^2}}\right) - \frac{1}{\pi}\arctan\left(\frac{C_\gamma s+1}{\sqrt{C_\gamma^2-1}\sqrt{1-s^2}}\right)$$

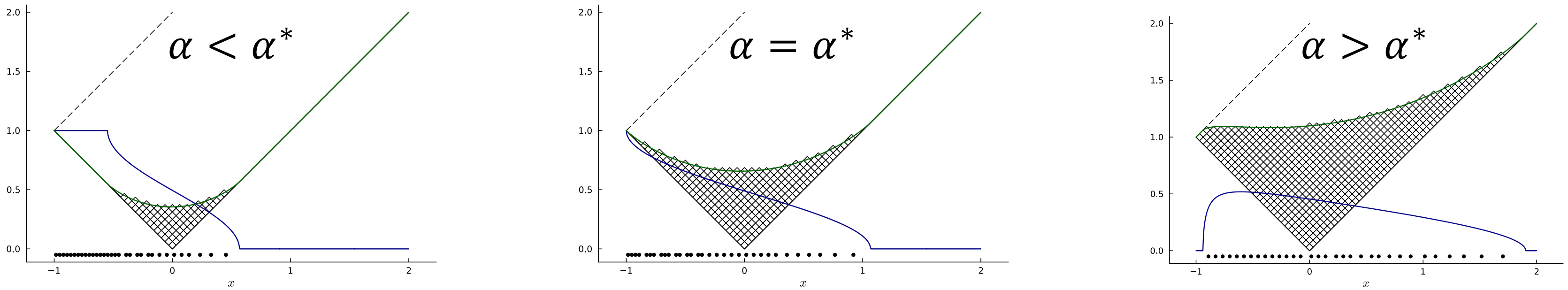
$$C_1 \equiv \frac{(2\gamma-1)\alpha+1}{2\sqrt{2\gamma-1}\alpha}, \quad C_\gamma \equiv \frac{2\gamma-1+\alpha}{2\sqrt{2\gamma-1}\alpha}$$



GWW相転移と
ボーズ・アインシュタイン凝縮

GWW phase transition in Young diagram

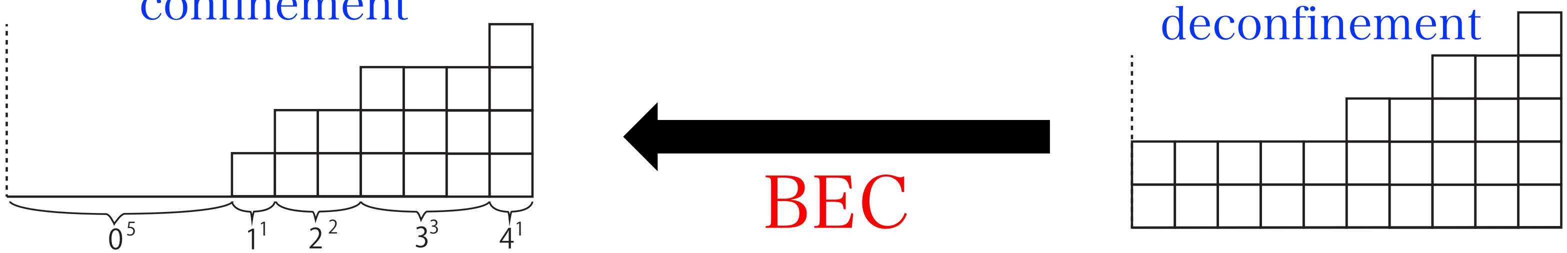
GWW phase transition takes place when the diagram touch to the bound



GWW phase transition in the bosonic picture

Young diagram: $\{0^{m_0} 1^{m_1} 2^{m_2} \cdots\}$

$(m_0 + m_1 + \cdots = N_c)$



$O(N_c)$ states occupy $E = 0$ in large N_c

cf) Hanada-Shimada-Wintergerst 2020
Watanabe-san's talk

Conclusion

- We rewrote the partition function of the FKM model on the cycle graph as a random partition.
- We reinterpreted the duality (functional equation) of the FKM model as the duality of the shape of the Young diagram.
- We determined the limiting shapes of the Young diagram in both phases and identified the GWW phase transition as a phenomenon that Young diagrams touch the boundary.
- We interpreted the GWW phase transition as the Bose-Einstein condensation.