Recent Developments in the Link Formulation of Twisted SUSY on a Lattice

離散的手法による場と時空のダイナミクス2025

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based on the collaboration with

A. D’Adda *INFN Torino, Italy, * passed away on April 23, 2022.*

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Contents of the talk

Most of the talk:

- Part 1: Gauge Covariant Link Formulation of Twisted N=D=4 and N=4 D=5 Super Yang-Mills on a Lattice

  D'Adda, Kawamoto, K.N. and Saito, arXiv:2412.19666 [hep-lat] to appear in JHEP
  K. N., Doctor Thesis, 2005, Hokkaido University

A bit about:

- Part 2: Non(anti)Commutative Superspace, BCH Closed Forms, and Dirac-Kähler Twisted Supersymmetry

  K.N., arXiv:2502.16410 [hep-th]

- Summary & Discussion

Three Main Ingredients to realize Exact SUSY on a Lattice

1. Dirac-Kähler Twisted SUSY (Marcus B-twist or Geometric Langlands twist)

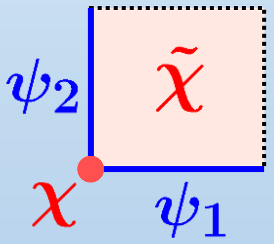
N=D=2: $$Q_{\alpha i} = (1Q + \gamma^\mu Q_\mu + \gamma^5 \tilde{Q})_{\alpha i}$$

N=D=4: $$Q_{\alpha i} = \frac{1}{\sqrt{2}}(1Q + \gamma^\mu Q_\mu + \gamma^{\mu\nu} Q_{\mu\nu} + \tilde{\gamma}^\mu \tilde{Q}_\mu + \gamma^5 \tilde{Q})_{\alpha i}$$

- solves Fermion doubling problem in terms of extended SUSY d.o.f.,

N=2,4 Twisted Fermion ⟺ Nf =2, 4 Staggered (Dirac-Kähler) Fermion

$\chi \ \psi_\mu \ \tilde{\chi}$



- also provides a solution of Lattice Leibniz rule, as we will see..

c.f. SUSY inv. is up to total derivative: $\delta_A S = \int d^D x \, \partial_\mu I^\mu_A = 0$

Need to use Leibniz rule: $\partial_\mu(fg) = (\partial_\mu f)g + f(\partial_\mu g)$

Three Main Ingredients to realize Exact SUSY on a Lattice

2. Supercovariant formulation with Link (anti)commutators and Jacobi identities

$$\nabla_A = D_A - i\Gamma_A(x,\theta_A), \qquad [\nabla_A, \{\nabla_B, \nabla_C\}] + cyclic = 0$$

・J. Wess and B. Zumino, Phys. Lett. B 66 (1977) 361. 42.
・R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B 133, 275 (1978).
・M. F. Sohnius, Nucl. Phys. B 136, 461 (1978). etc..

$$\boxed{(\nabla_A)_{x+a_A,x}, \qquad [\nabla_A, \{\nabla_B, \nabla_C\}]_{x+a_A+a_B+a_C,x} + cyclic = 0}$$

provides manifest gauge covariance at every step of calculation.

3. Group and Algebraic structure with Non-(anti)commutative Grassmann coordinates θ_A and parameters ξ_A .

SUSY on a Lattice ?

- Motivated by non-perturbative study of SUSY theories, numerical calculations of Duality and Gauge/Gravity correspondence, etc..
- Needs understandings of the SUSY preserving discrete spacetime through super Lie Group and Algebra.

Continuum spacetime:

$$\{Q_A, Q_B\} = f^{\mu}_{AB}\, \partial_\mu$$

Algebraic elements of super Poincaré

$\Rightarrow$

On the Lattice: ?

difference op.

$$\{Q_A, Q_B\} = f^{\mu}_{AB}\, \Delta_\mu \sim f^{\mu}_{AB}\, e^{\partial_\mu}$$

Algebraic elements

Group element defining discrete spacetime

Looks absurd ! But it's possible with Dirac-Kähler Twisted SUSY Algebra.

Part 1: Warming-up: N=D=2 Twisted SUSY Algebra

A. D'Adda, I. Kanamori, N. Kawamoto and K. N, Nucl.Phys. B707 (2005) 100-144

$$\{Q^i_\alpha, Q^j_\beta\} = 2\delta^{ij}(\gamma^\mu)_{\alpha\beta}\partial_\mu$$

$$\begin{pmatrix} \mu,\nu = 1,2 : \text{2D Euclidean} \\ \alpha,\beta = 1,2 : \text{spinor indices} \\ i,j = 1,2 : \text{internal indices} \\ \gamma^1 = \sigma_3,\ \gamma^2 = \sigma_1,\ \gamma^5 = \gamma^1\gamma^2 \end{pmatrix}$$

Dirac-Kähler Twist

$$Q^i_\alpha = (1Q + \gamma^\mu Q_\mu + \gamma^5 \tilde{Q})^i_\alpha$$

Q : Fermionic Scalar

Q_μ : Fermionic Vector

$\tilde{Q}$: Fermionic Pseudo Scalar

$$\{Q, Q_\mu\} = +i\partial_\mu$$
$$\{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\partial_\nu$$

On the Lattice ?

$$\{Q, Q_\mu\} = +i\Delta_{\pm\mu}$$
$$\{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{\pm\nu}$$

$\Delta_{\pm\mu}$: difference operators

Discretization of Twisted SUSY Algebra

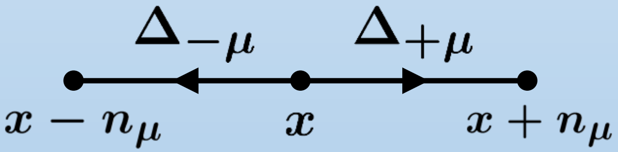
$$\boxed{\{\underline{Q_A, Q_B}\} = f^{\mu}_{AB}\underline{\Delta_{\pm\mu}}} \qquad Q_A = (Q, Q_\mu, \tilde{Q})$$

R.H.S.

$$\boxed{\begin{aligned}(\Delta_{\pm\mu}\Phi)(x) &= \pm[\Phi(x \pm n_\mu) - \Phi(x)] \\ &\equiv \Delta_{\pm\mu}\Phi(x) - \Phi(x \pm n_\mu)\Delta_{\pm\mu}\end{aligned}}$$

Defined as "Shifted" Commutator with

$$\Delta_{\pm\mu} = (\Delta_{\pm\mu})_{x\pm n_\mu, x} = \mp 1$$



Lattice Leibniz rule for $\Delta_{\pm\mu}$:

$$\Delta_{\pm\mu}(\Phi(x)\Psi(x)) = (\Delta_{\pm\mu}\Phi(x))\Psi(x) + \Phi(x \pm n_\mu)(\Delta_{\pm\mu}\Psi(x))$$

as well known

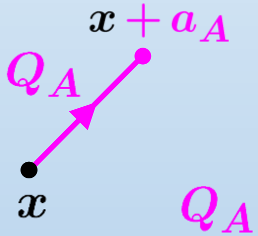
Discretization of Twisted SUSY Algebra

$$\boxed{\{Q_A, Q_B\} = f^{\mu}_{AB}\Delta_{\pm\mu}} \qquad Q_A = (Q, Q_\mu, \tilde{Q})$$

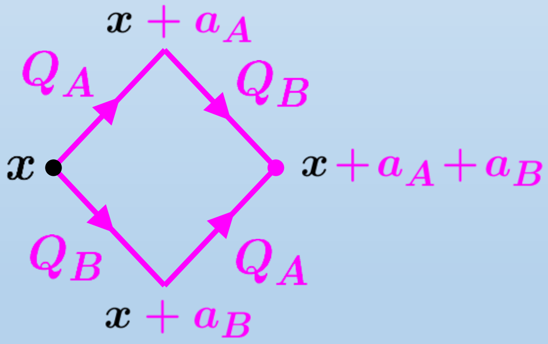
L.H.S. $\quad (Q_A\Phi)(x) \equiv Q_A\Phi(x) \pm \Phi(x+a_A)Q_A$

Introduce "Shifted" (Anti-) Commutator with Link Supercharge:

$$\Downarrow \quad Q_A = (Q_A)_{x+a_A,x}$$



$$\boxed{\begin{aligned}(\{Q_A, Q_B\}\Phi)(x) &= \{Q_A, Q_B\}\Phi(x) \\ &\quad -\Phi(x+a_A+a_B)\{Q_A, Q_B\}\end{aligned}}$$



$$\begin{aligned}\{Q_A, Q_B\}(\Phi(x)\Psi(x)) &= (\{Q_A, Q_B\}\Phi(x))\Psi(x) \\ &\quad +\Phi(x+a_A+a_B)(\{Q_A, Q_B\}\Psi(x))\end{aligned}$$

Consistency Requires: ⇒

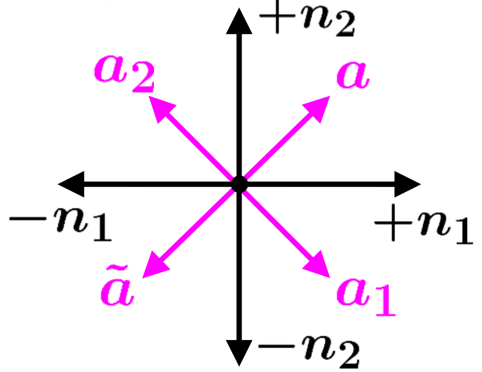
Lattice Leibniz rule condition

$$a_A + a_B = +n_\mu \quad \text{for} \quad \Delta_{+\mu}$$
$$a_A + a_B = -n_\mu \quad \text{for} \quad \Delta_{-\mu}$$

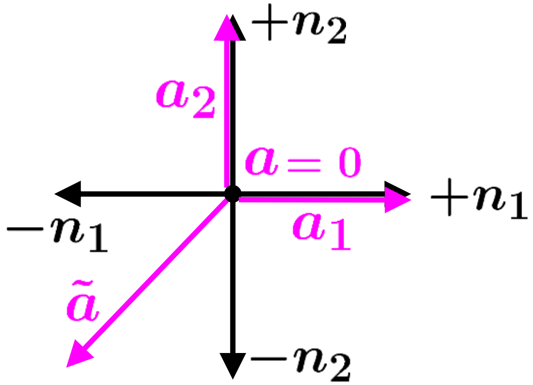
Solution exists for Twisted SUSY

$$a + a_\mu = +n_\mu$$
$$\tilde{a} + a_\mu = -|\epsilon_{\mu\nu}| n_\nu$$
$$a + a_1 + a_2 + \tilde{a} = 0$$

- Symm. Choice



- Asymm. Choice



Twisted N=D=2
Lattice SUSY Algebra

$$\{Q, Q_\mu\} = +i\Delta_{+\mu}$$
$$\{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu}$$

N=D=4 Twisted SUSY Algebra

D'Adda, Kawamoto, K.N. and Saito,
arXiv:2412.19666 [hep-lat] to appear in JHEP

$$\{Q_\alpha^{\ i}, \overline{Q}_\beta^{\ j}\} = 2\delta^{ij}(\gamma^\mu)_{\alpha\beta}\partial_\mu$$

$$\begin{pmatrix} \mu,\nu = 1 \sim 4 : \text{4D Euclidean} \\ \alpha,\beta = 1 \sim 4 : \text{spinor indices} \\ i,j = 1 \sim 4 : \text{internal indices} \\ C = \gamma_2\gamma_4 : \gamma_\mu^T = C^{-1}\gamma_\mu C \end{pmatrix}$$

$$\bar{Q}_\alpha^i = (C^{-1}QC)_\alpha^i$$

Dirac-Kähler Twist

$$Q_{\alpha i} = (1Q + \gamma^\mu Q_\mu + \gamma^{\mu\nu} Q_{\mu\nu} + \tilde{\gamma}^\mu \tilde{Q}_\mu + \gamma^5 \tilde{Q})_{i\alpha}$$

$$\begin{pmatrix} \gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \\ \tilde{\gamma}^\mu \equiv \gamma_\mu\gamma_5 \\ \gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4 \end{pmatrix}$$

$$\begin{aligned} \{Q, Q_\mu\} &= -i\partial_\mu \\ \{Q_{\rho\sigma}, Q_\mu\} &= +i\delta_{\rho\sigma\mu\nu}\partial_\nu \\ \{Q_{\rho\sigma}, \tilde{Q}_\mu\} &= -i\epsilon_{\rho\sigma\mu\nu}\partial_\nu \\ \{\tilde{Q}, \tilde{Q}_\mu\} &= -i\partial_\mu \end{aligned}
\quad \Rightarrow \quad
\begin{aligned} \{Q, Q_\mu\} &= -i\Delta_{\pm\mu} \\ \{Q_{\rho\sigma}, Q_\mu\} &= +i\delta_{\rho\sigma\mu\nu}\Delta_{\pm\nu} \\ \{Q_{\rho\sigma}, \tilde{Q}_\mu\} &= -i\epsilon_{\rho\sigma\mu\nu}\Delta_{\pm\nu} \\ \{\tilde{Q}, \tilde{Q}_\mu\} &= -i\Delta_{\pm\mu} \end{aligned}$$

$$\delta_{\rho\sigma\mu\nu} \equiv \delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\sigma\mu}$$

On the Lattice ?

Twisted N=D=4 Lattice SUSY Algebra

$$\{Q, Q_\mu\} = -i\Delta_{+\mu}$$
$$\{Q_{\rho\sigma}, Q_\mu\} = +i\delta_{\rho\sigma\mu\nu}\Delta_{-\nu}$$
$$\{Q_{\rho\sigma}, \tilde{Q}_\mu\} = -i\epsilon_{\rho\sigma\mu\nu}\Delta_{+\nu}$$
$$\{\tilde{Q}, \tilde{Q}_\mu\} = -i\Delta_{-\mu}$$

$\Leftarrow$

Lattice Leibniz rule cond.

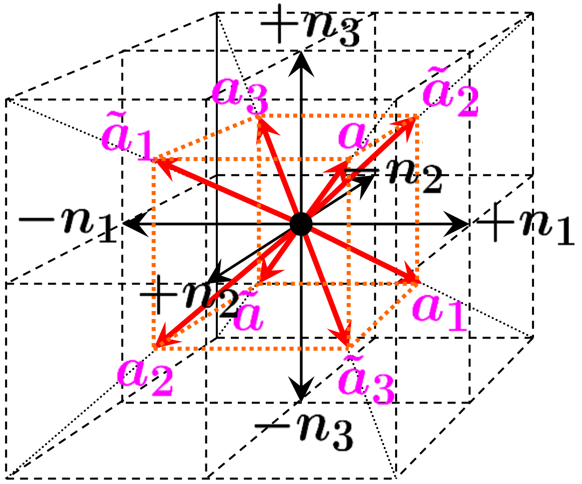
$$a + a_\mu = +n_\mu$$
$$a_{\rho\sigma} + a_\mu = -|\delta_{\rho\sigma\mu\nu}| n_\nu$$
$$a_{\rho\sigma} + \tilde{a}_\mu = +|\epsilon_{\rho\sigma\mu\nu}| n_\nu$$
$$\tilde{a} + \tilde{a}_\mu = -n_\mu$$

$$\sum a_A = 0$$

Symmetric choice

	n_1	n_2	n_3	n_4
a	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
a_{12}	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
a_{13}	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
a_{14}	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
a_{23}	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
a_{24}	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
a_{34}	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\tilde{a}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

	n_1	n_2	n_3	n_4
a_1	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
a_2	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
a_3	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
a_4	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$\tilde{a}_4$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
$\tilde{a}_3$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$\tilde{a}_2$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
$\tilde{a}_1$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$



$a_A \sim (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}), \quad 2^4 = \underline{16}$ $\Leftarrow$ # of vertices of 4D Hypercube

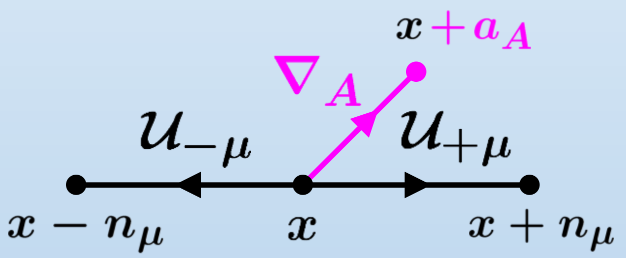
Guage Covariant extension to N=D=4 Twisted SYM

Introduce Bosonic & Fermionic Link variables

$$
\boxed{
\begin{aligned}
(\Delta_{\pm\mu})_{x\pm n_\mu,x} &\rightarrow \mp(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x} \\
(Q_A)_{x+a_A,x} &\rightarrow (\nabla_A)_{x+a_A,x}
\end{aligned}
}
$$

Gauge trans.

$$
\begin{aligned}
(\mathcal{U}_{\pm\mu})' &= G_{x\pm n_\mu}\,(\mathcal{U}_{\pm\mu})\,G_x^{-1} \\
(\nabla_A)' &= G_{x+a_A}\,(\nabla_A)\,G_x^{-1}
\end{aligned}
$$



- $(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x} = (e^{\pm i(A_\mu \pm i V_\mu)})_{x\pm n_\mu,x},$

  $V_\mu \ (\mu = 1 \sim 4)$: Twisted scalar (vector) fields in SYM multiplet

- $\mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \neq 1$

Expected multiplet of N=D=4 Lattice SYM

Twisting of N=D=4

Define $SO(4)^{twisted}_{Lorentz}$ as the diagonal subgroup of $SO(4)_{Lorents} \otimes SO(4)_{Internal}$

- J.P. Yamron, Phys.Lett. B213 (1988) 325
- C. Vafa, E. Witten, Nucl.Phys. B431 (1994) 3
- N. Marcus, Nucl. Phys. B452 (1995) 331

$\vdots$

Dirac-Kähler Twisting

helicity	**Untwisted** $(SU(2)_L, SU(2)_R, SO(4)_I)$	**A-type** $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1})$	**B-type** $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$
$+1$	$\omega_\mu(\mathbf{2},\mathbf{2},\mathbf{1})$	$(\mathbf{2},\mathbf{2})$	$(\mathbf{2},\mathbf{2})$
$+\frac{1}{2}$	$\lambda^\alpha_i(\mathbf{2},\mathbf{1},\mathbf{4})$	$2(\mathbf{1},\mathbf{1}) \oplus 2(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2})$
0	$\phi_{ij}(\mathbf{1},\mathbf{1},\mathbf{6})$	$3(\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1})$	$\boxed{2(\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2})}$
$-\frac{1}{2}$	$\bar\lambda_{\dot\alpha i}(\mathbf{1},\mathbf{2},\mathbf{4})$	$2(\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{2})$
-1	$\omega_\mu(\mathbf{2},\mathbf{2},\mathbf{1})$	$(\mathbf{2},\mathbf{2})$	$(\mathbf{2},\mathbf{2})$

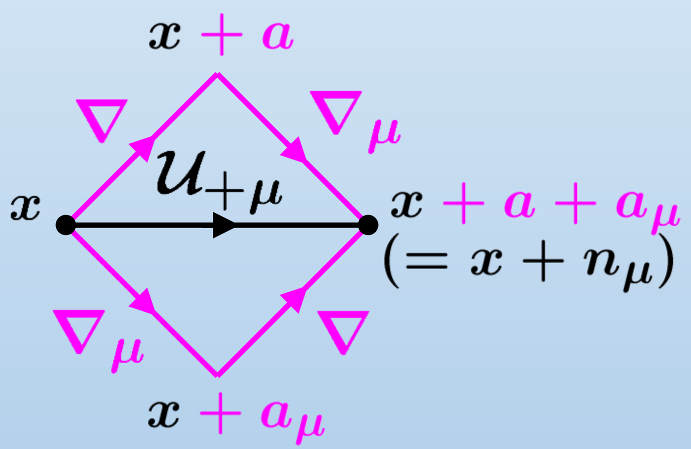
- $2(\mathbf{1},\mathbf{1}) \cdots W,\ F$: 2 scalars
- $(\mathbf{2},\mathbf{2}) \cdots V_\mu$: 4-vector (twisted scalars)

N=D=4 SYM constraints on a Lattice

$$
\begin{aligned}
\{\nabla, \nabla_\mu\}_{x+a+a_\mu,x} &= +i(\mathcal{U}_{+\mu})_{x+n_\mu,x}, \\
\{\nabla_{\rho\sigma}, \nabla_\mu\}_{x+a_{\rho\sigma}+a_\mu,x} &= +i\delta_{\rho\sigma\mu\nu}(\mathcal{U}_{-\nu})_{x-n_\nu,x}, \\
\{\nabla_{\rho\sigma}, \tilde{\nabla}_\mu\}_{x+a_{\rho\sigma}+\tilde{a}_\mu,x} &= +i\epsilon_{\rho\sigma\mu\nu}(\mathcal{U}_{+\nu})_{x+n_\nu,x}, \\
\{\tilde{\nabla}, \tilde{\nabla}_\mu\}_{x+\tilde{a}+\tilde{a}_\mu,x} &= -i(\mathcal{U}_{-\mu})_{x-n_\mu,x}, \\
\\
\{\nabla, \tilde{\nabla}\}_{x+a+\tilde{a},x} &= -i(W)_{x+a+\tilde{a},x}, \\
\{\nabla_{\mu\nu}, \nabla_{\rho\sigma}\}_{x+a_{\mu\nu}+a_{\rho\sigma},x} &= +i\epsilon_{\mu\nu\rho\sigma}(W)_{x+a_{\mu\nu}+a_{\rho\sigma},x}, \\
\{\nabla_\mu, \tilde{\nabla}_\nu\}_{x+a_\mu+\tilde{a}_\nu,x} &= -i\delta_{\mu\nu}(F)_{x+a_\mu+\tilde{a}_\nu,x}, \\
\{others\} &= 0,
\end{aligned}
$$

"Shifted" Anti-commutator

$$
\begin{aligned}
&\{\nabla, \nabla_\mu\}_{x+a+a_\mu,x} \\
&\equiv (\nabla)_{x+a+a_\mu,x+a_\mu}(\nabla_\mu)_{x+a_\mu,x} \\
&\quad +(\nabla_\mu)_{x+a+a_\mu,x+a}(\nabla)_{x+a,x}
\end{aligned}
$$



Lattice Leibniz rule condition becomes gauge covariant condition on the lattice.

$$
\left(\because\ a + a_\mu = +n_\mu \quad \vdots \right)
$$

Jacobi Identity analysis & Lattice multiplet

$$[\nabla_\mu\{\nabla_\nu,\nabla\}]_{x+a_\mu+n_\nu,x} + (cyclic) = 0,$$

: trivially holds even for shifted (anti)commutators

$$\Downarrow$$

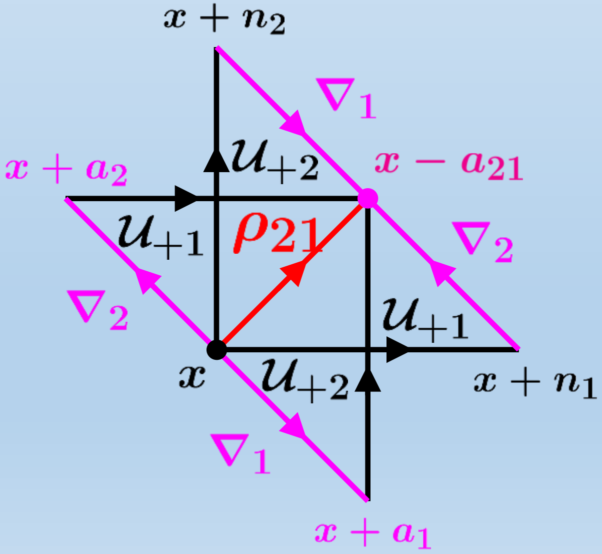
$$[\nabla_\mu,\mathcal{U}_{+\nu}]_{x+a_\mu+n_\nu,x} + [\nabla_\nu,\mathcal{U}_{+\mu}]_{x+a_\nu+n_\mu,x} = 0,$$

$$\vdots$$

Define fermionic link components

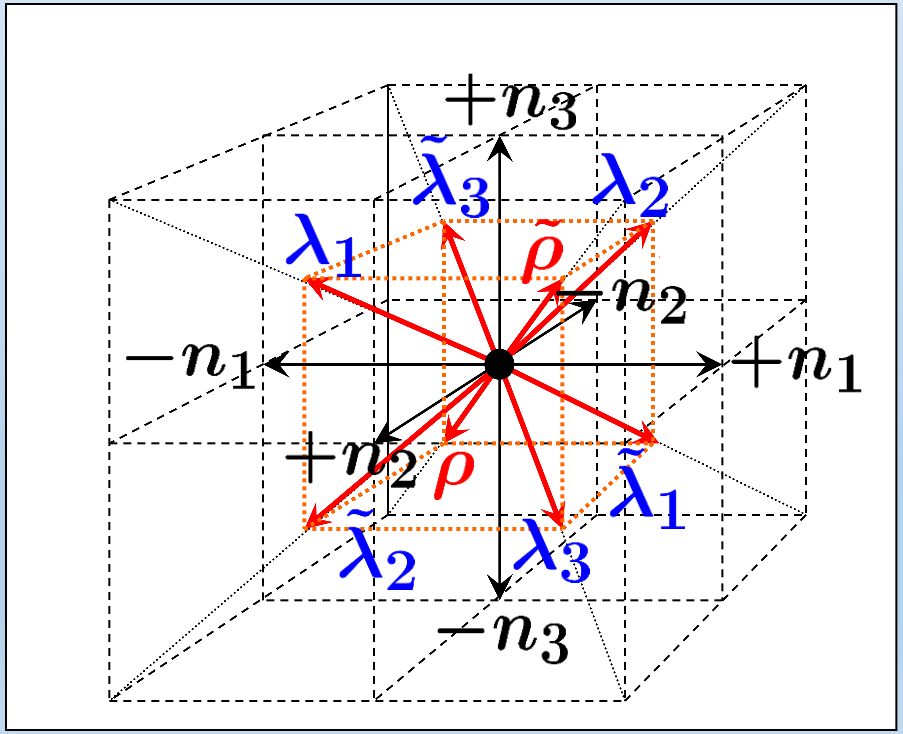
$$[\nabla_\mu,\mathcal{U}_{+\nu}]_{x+a_\mu+n_\nu,x} \equiv -(\rho_{\mu\nu})_{x-a_{\mu\nu},x}$$

$$\vdots$$



$$(\rho,\lambda_\mu,\rho_{\mu\nu},\tilde{\lambda}_\mu,\tilde{\rho}):$$

N=D=4 Twisted Lattice Fermions



N=D=4 SUSY trans. laws

	s	$s_{\rho\sigma}$	$\tilde{s}$
$\mathcal{U}_{+\mu}$	0	$-\delta_{\rho\sigma\mu\nu}\lambda_\nu$	$+\tilde{\lambda}_\mu$
$\mathcal{U}_{-\mu}$	$+\lambda_\mu$	$+\epsilon_{\rho\sigma\mu\nu}\tilde{\lambda}_\nu$	0
W	0	0	0
F	$-\tilde{\rho}$	$-\frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}\rho_{\alpha\beta}$	$-\rho$
ρ	$+\frac{i}{2}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]+[W,F])$	$-i[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]$	0
λ_μ	0	$-i\epsilon_{\rho\sigma\mu\nu}[\mathcal{U}_{+\nu},W]$	$+i[\mathcal{U}_{-\mu},W]$
$\rho_{\mu\nu}$	$+i[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]$	$+i\delta_{\rho\sigma\mu\lambda}[\mathcal{U}_{+\nu},\mathcal{U}_{-\lambda}]-i\delta_{\rho\sigma\nu\lambda}[\mathcal{U}_{+\mu},\mathcal{U}_{-\lambda}]$ $-\frac{i}{2}\delta_{\rho\sigma\mu\nu}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]+[W,F])$	$+\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}[\mathcal{U}_{-\alpha},\mathcal{U}_{-\beta}]$
$\tilde{\lambda}_\mu$	$-i[\mathcal{U}_{+\mu},W]$	$+i\delta_{\rho\sigma\mu\nu}[\mathcal{U}_{-\nu},W]$	0
$\tilde{\rho}$	0	$+\frac{i}{2}\epsilon_{\rho\sigma\alpha\beta}[\mathcal{U}_{+\alpha},\mathcal{U}_{+\beta}]$	$-\frac{i}{2}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]-[W,F])$

	s_ρ	$\tilde{s}_\rho$
$\mathcal{U}_{+\mu}$	$-\rho_{\rho\mu}$	$-\delta_{\rho\mu}\tilde{\rho}$
$\mathcal{U}_{-\mu}$	$+\delta_{\rho\mu}\rho$	$-\frac{1}{2}\epsilon_{\rho\mu\alpha\beta}\rho_{\alpha\beta}$
W	$+\tilde{\lambda}_\rho$	$-\lambda_\rho$
F	0	0
ρ	0	$+i[\mathcal{U}_{-\rho},F]$
λ_μ	$+i[\mathcal{U}_{+\rho},\mathcal{U}_{-\mu}]$ $-\frac{i}{2}\delta_{\rho\mu}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]+[W,F])$	$-\frac{i}{2}\epsilon_{\rho\mu\alpha\beta}[\mathcal{U}_{+\alpha},\mathcal{U}_{+\beta}]$
$\rho_{\mu\nu}$	$-i\epsilon_{\rho\sigma\mu\nu}[\mathcal{U}_{-\sigma},F]$	$-i\delta_{\rho\sigma\mu\nu}[\mathcal{U}_{+\sigma},F]$
$\tilde{\lambda}_\mu$	$+\frac{i}{2}\epsilon_{\rho\mu\alpha\beta}[\mathcal{U}_{-\alpha},\mathcal{U}_{-\beta}]$	$+i[\mathcal{U}_{+\mu},\mathcal{U}_{-\rho}]$ $-\frac{i}{2}\delta_{\rho\mu}([\mathcal{U}_{+\lambda},\mathcal{U}_{-\lambda}]-[W,F])$
$\tilde{\rho}$	$-i[\mathcal{U}_{+\rho},F]$	0

$$s_A(\varphi)_{x+a_\varphi,x}$$
$$\equiv [\nabla_A,\varphi\}_{x+a_\varphi+a_A,x}$$
$$\varphi = (\mathcal{U}_{\pm\mu}, W, F, \rho, \lambda_\mu, \rho_{\mu\nu}, \tilde{\lambda}_\mu, \tilde{\rho})$$

No Auxiliary fields
in the multiplet

Resulting SUSY Algebra closes only on-shell

$$
\begin{aligned}
\{s, s_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq +i[\mathcal{U}_{+\mu}, \varphi]_{x+n_\mu+a_\varphi,x} \\
\{s_{\rho\sigma}, s_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq +i\delta_{\rho\sigma\mu\nu}[\mathcal{U}_{-\nu}, \varphi]_{x-n_\nu+a_\varphi,x} \\
\{s_{\rho\sigma}, \tilde{s}_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq +i\epsilon_{\rho\sigma\mu\nu}[\mathcal{U}_{+\nu}, \varphi]_{x+n_\nu+a_\varphi,x} \\
\{\tilde{s}, \tilde{s}_\mu\}(\varphi)_{x+a_\varphi,x} &\doteq -i[\mathcal{U}_{-\mu}, \varphi]_{x-n_\mu+a_\varphi,x} \\
\\
\{s, \tilde{s}\}(\varphi)_{x+a_\varphi,x} &\doteq -i[W, \varphi]_{x+a+\tilde{a}+a_\varphi,x} \\
\{s_{\mu\nu}, s_{\rho\sigma}\}(\varphi)_{x+a_\varphi,x} &\doteq +i\epsilon_{\mu\nu\rho\sigma}[W, \varphi]_{x+a+\tilde{a}+a_\varphi,x} \\
\{s_\mu, \tilde{s}_\nu\}(\varphi)_{x+a_\varphi,x} &\doteq -i\delta_{\mu\nu}[F, \varphi]_{x-a-\tilde{a}+a_\varphi,x} \\
\{others\}(\varphi)_{x+a_\varphi,x} &\doteq 0
\end{aligned}
$$

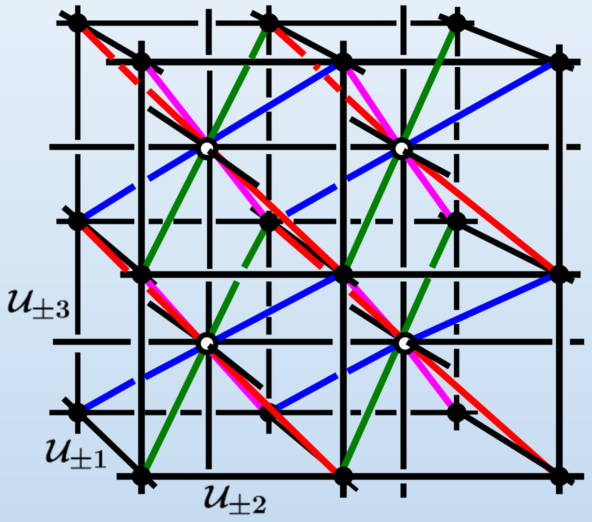
$$\varphi : (\mathcal{U}_{\pm\mu}, W, F, \rho, \lambda_\mu, \rho_{\mu\nu}, \tilde{\lambda}_\mu, \tilde{\rho})$$

Equalities hold up to eqns. of motion:

$$
\begin{aligned}
[\mathcal{U}_{+\mu}, \lambda_\mu] - [W, \tilde{\rho}] &= 0 \\
[\mathcal{U}_{-\mu}, \tilde{\lambda}_\mu] - [W, \rho] &= 0 \\
[\mathcal{U}_{+\mu}, \rho] - [\mathcal{U}_{-\nu}, \rho_{\mu\nu}] + [F, \tilde{\lambda}_\mu] &= 0 \\
[\mathcal{U}_{-\mu}, \tilde{\rho}] + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[\mathcal{U}_{+\nu}, \rho_{\rho\sigma}] + [F, \lambda_\mu] &= 0 \\
\delta_{\mu\nu\rho\sigma}[\mathcal{U}_{-\rho}, \lambda_{-\sigma}] + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[W, \rho_{\rho\sigma}] - \epsilon_{\mu\nu\rho\sigma}[\mathcal{U}_{+\rho}, \tilde{\lambda}_\sigma] &= 0
\end{aligned}
$$

Lattice N=D=4 Dirac-Kähler Twisted SYM Action

$$
\begin{aligned}
S_{TSYM}^{N=D=4} = \sum_x \mathrm{tr}\Big[& -\frac{1}{2}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu}[\mathcal{U}_{-\mu},\mathcal{U}_{-\nu}]_{x-n_\mu-n_\nu,x} \\
& +\frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + \frac{1}{4}[W,F]_{x,x}[W,F]_{x,x} \\
& -\frac{1}{2}[\mathcal{U}_{+\mu},W]_{x,x-n_\mu-a-\tilde{a}}[\mathcal{U}_{-\mu},F]_{x-n_\mu-a-\tilde{a},x} \\
& -\frac{1}{2}[\mathcal{U}_{-\mu},W]_{x,x+n_\mu-a-\tilde{a}}[\mathcal{U}_{+\mu},F]_{x+n_\mu-a-\tilde{a},x} \\
& +i(\lambda_\mu)_{x,x+a_\mu}[\mathcal{U}_{+\mu},\rho]_{x+a_\mu,x} - i\tilde{\rho}_{x,x+\tilde{a}}[W,\rho]_{x+\tilde{a},x} + i(\lambda_\mu)_{x,x+a_\mu}[F,\tilde{\lambda}_\mu]_{x+a_\mu,x} \\
& -i(\lambda_\mu)_{x,x+a_\mu}[\mathcal{U}_{-\nu},\rho_{\mu\nu}]_{x+a_\mu,x} + i(\tilde{\rho})_{x,x+\tilde{a}}[\mathcal{U}_{-\mu},\tilde{\lambda}_\mu]_{x+\tilde{a},x} \\
& +\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}(\tilde{\lambda}_\mu)_{x,x+\tilde{a}_\mu}[\mathcal{U}_{+\nu},\rho_{\rho\sigma}]_{x+\tilde{a}_\mu,x} + \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}(\rho_{\mu\nu})_{x,x+a_{\mu\nu}}[W,\rho_{\rho\sigma}]_{x+a_{\mu\nu},x}\Big]
\end{aligned}
$$



— $(\rho, \tilde{\rho})$
— $(\lambda_3, \tilde{\lambda}_3)$
— $(\lambda_2, \tilde{\lambda}_2)$
— $(\lambda_1, \tilde{\lambda}_1)$

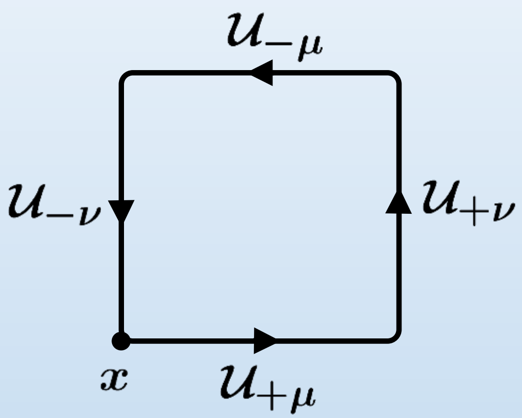
For Symm. Choice

$$\sum_x = \sum_{\bullet} + \sum_{\circ}$$

- ● Integer sites (m_1, m_2, m_3, m_4)
- ○ Half-Integer sites $(m_1+\frac{1}{2}, m_2+\frac{1}{2}, m_3+\frac{1}{2}, m_4+\frac{1}{2})$

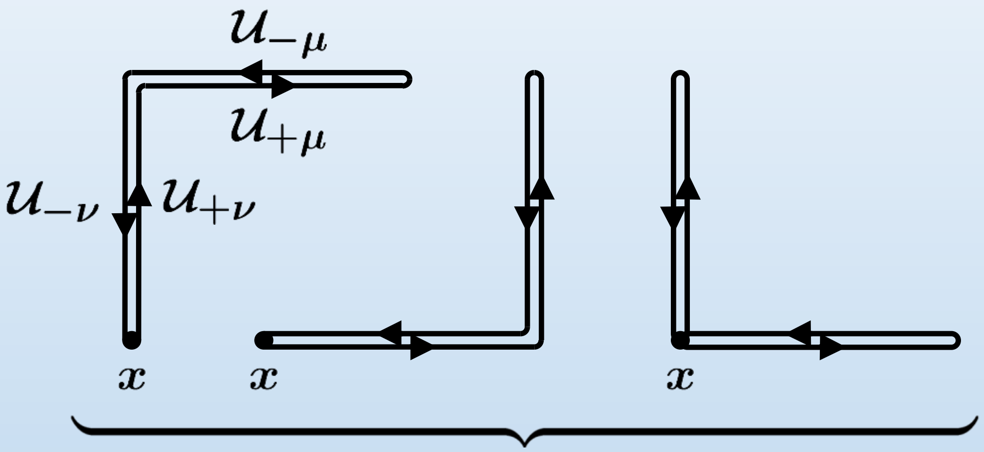
Boson terms in the Action

Plaquette



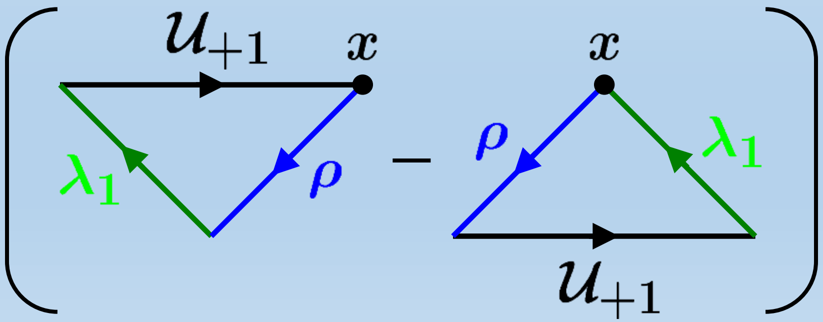
Zero-area loops

$(\mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \neq 1)$



Contribution of Scalar Fields

Fermion terms in the Action



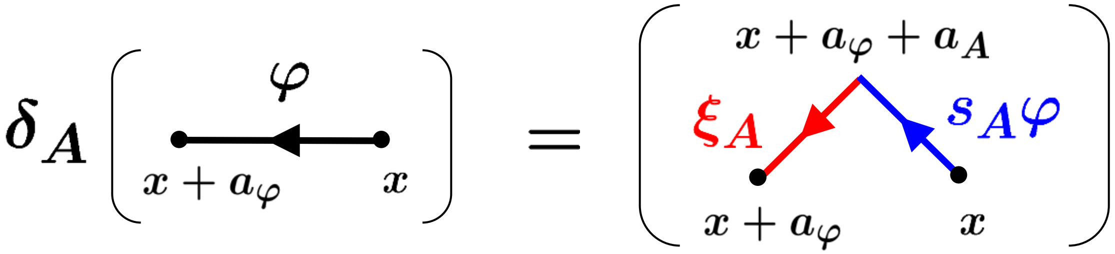
etc..

Remark 1: Gauge Covariant SUSY variations

REPLY to: F. Bruckmann, S. Catterall and M. de Kok, Phys. Rev. D 75, 045016 (2007)

Gauge cov. SUSY variation for φ :

$$\delta_A(\varphi)_{x+a_\varphi,x} = (\xi_A)_{x+a_\varphi,x+a_\varphi+a_A}\,(s_A\varphi)_{x+a_\varphi+a_A,x}$$



ξ_A : Grassmann link parameters satisfying:

- $\{\nabla_A, \xi_B\}_{x+a_A-a_B,x} = 0$
- $(\xi_A)_{x+a_A,x} \to G^{-1}_{x+a_A}\,(\xi_A)_{x+a_A,x}\,G_x$

$$S[\varphi+\delta_A\varphi] - S[\varphi] = 0$$: SUSY inv. of Action can be shown

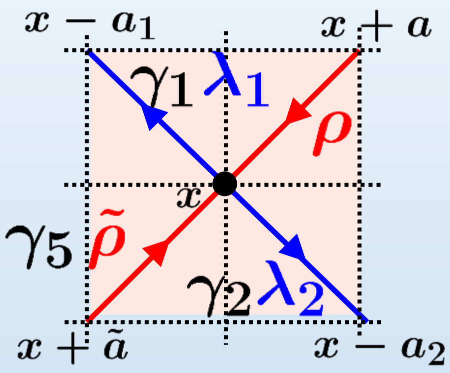
for all the supercharges: $\delta_A = (\xi s,\ \xi_\mu s_\mu,\ \xi_{\mu\nu} s_{\mu\nu},\ \tilde{\xi}_\mu \tilde{s}_\mu,\ \tilde{\xi}\tilde{s})$: μ, ν no sum

by utilizing cyclic permutation properties under $\sum_x \mathrm{tr}$ and Jacobi identities.

Remark 2: Staggered Structure of Twisted Fermions

Free part of (for simplicity N=D=2 case)

$$\sum_x \mathrm{Tr}\Big[-i[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x-a}(\rho)_{x-a,x} \\ -i(\tilde{\rho})_{x,x+\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu},\lambda_\nu]_{x+\tilde{a},x}\Big]$$



$$\zeta_{\alpha i}(x)=\frac{1}{2}\Big((\rho)_{x,x+a}+\gamma_\mu(\lambda_\mu)_{x-a_\mu,x}+\gamma_5(\tilde{\rho})_{x,x+\tilde{a}}\Big)_{\alpha i}$$

$$\sum_x\Big[-i\bar{\zeta}_{i\alpha}(x)(\gamma_\mu)_{\alpha\beta}\frac{\Delta_{+\mu}+\Delta_{-\mu}}{2}\zeta_{\beta i}(x) \\ -i\bar{\zeta}_{i\alpha}(x)(\gamma_5)_{\alpha\beta}\frac{\Delta_{+\mu}-\Delta_{-\mu}}{2}\zeta_{\beta i}(x)(\gamma_5\gamma_\mu)_{ji}\Big]$$

1st Diff. term w.r.t. Double size lattice

2nd Diff. term ~ O(lat.const.)

N=2,4 Twisted Fermions ⟺ Nf = 2,4 Staggered Fermions

Naïve Continuum limit

$$
\begin{aligned}
(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x} &= (e^{\pm i(A_\mu \pm iV_\mu)})_{x\pm n_\mu,x} \\
&= (1 \pm i(A_\mu \pm iV_\mu) + \cdots)_{x\pm n_\mu,x}
\end{aligned}
$$

$$
\begin{aligned}
S_{TSYM}^{N=D=4} \to S_{cont} = \int d^4x \ \mathrm{tr} \Bigg[& \frac{1}{2}F_{\mu\nu}F_{\mu\nu} + [\mathcal{D}_\mu, W][\mathcal{D}_\mu, F] + [\mathcal{D}_\mu, V_\nu][\mathcal{D}_\mu, V_\nu] \\
& -\frac{1}{2}[V_\mu, V_\nu][V_\mu, V_\nu] + [V_\mu, W][V_\mu, F] + \frac{1}{4}[W, F][W, F] \\
& -i\lambda_\mu[\mathcal{D}_\mu, \rho] - i\lambda_\mu[V_\mu, \rho] - i\tilde{\rho}[W, \rho] \\
& -i\lambda_\mu[\mathcal{D}_\nu, \rho_{\mu\nu}] + i\lambda_\mu[V_\nu, \rho_{\mu\nu}] \\
& +i\tilde{\rho}[\mathcal{D}_\mu, \tilde{\lambda}_\mu] - i\tilde{\rho}[V_\mu, \tilde{\lambda}_\mu] + i\lambda_\mu[F, \tilde{\lambda}_\mu] \\
& -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\mu[\mathcal{D}_\nu, \rho_{\rho\sigma}] - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\mu[V_\nu, \rho_{\rho\sigma}] + \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}\rho_{\mu\nu}[W, \rho_{\rho\sigma}] \Bigg]
\end{aligned}
$$

coincides with B-type twisted N=D=4 SYM

Short Summary of Link Formulation

- N=D=4 Twisted SYM on a Lattice is constructed with:
  - Lattice SUSY Algebra for All Supercharges
  - SUSY inv. Action w.r.t. All Supercharges

    with Grassmann link parameter ξ_A

- Why Twisted SUSY ?
  - Dirac-Kähler (Simplicial) Structure of Fermions

    Twisted Fermions $\Longleftrightarrow$ Staggered Fermions

    (N-extended SUSY) (N-flavors)

  - provides manifest gauge covariance via Lattice Leibniz rule

Group and Algebraic aspect of the formulation

We show that an Exponentiation of Bosonic super-covariant derivatives $\boldsymbol{\nabla}_{\pm\boldsymbol{\mu}}$ lead to Link (anti)commutator formulation.

⇨ Microscopic understanding of Lattice SUSY.

Begin with the super-covariant constraint in continuum spacetime.

$$\boxed{\{\nabla_A, \nabla_B\} \;=\; f^{\mu}_{AB}\nabla_{\pm\mu}}$$

: locally gauge covariant $(\nabla_A, \nabla_B, \nabla_{\pm\mu}) \to \mathcal{G}^{-1}(x)(\nabla_A, \nabla_B, \nabla_{\pm\mu})\,\mathcal{G}(x)$

where ∇_A and ∇_B : fermionic super-covariant derivatives $(\nabla, \nabla_\mu, \nabla_{\rho\sigma}, \tilde{\nabla}_\mu, \tilde{\nabla})$

$$\begin{aligned} \nabla_{\pm\mu} &= \partial_\mu - i\Gamma^{\pm}_{\mu} \\ &= \partial_\mu - i(\omega_\mu \pm iV_\mu) + \cdots \end{aligned}$$

: bosonic super-covariant derivatives

- ω_μ : gauge fields
- V_μ : twisted scalars

Group and Algebraic aspect of the formulation

Promote the Bosonic super-covariant derivative $\boldsymbol{\nabla_{\pm\mu}}$ to exponentiated Group element.

$$\{\nabla_A, \nabla_B\} = -f^{\mu}_{AB}\, e^{-\nabla_{+\mu}},$$

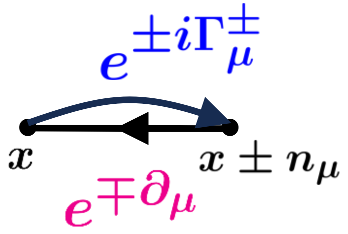
$$\{\nabla_A, \nabla_B\} = +f^{\mu}_{AB}\, e^{+\nabla_{-\mu}},$$

- f^{μ}_{AB} : numerical coefficient

Note that both hand sides still locally gauge covariant :

$$e^{\mp\nabla_{\pm\mu}} \to \mathcal{G}^{-1}(x)\, e^{\mp\nabla_{\pm\mu}}\, \mathcal{G}(x)$$

Why the r.h.s. are still local ?



$e^{\mp\nabla_{\pm\mu}}$ contain contributions of :

- $\pm i\Gamma^{\pm}_{\mu}$ connecting x to $x \pm n_{\mu}$ in a gauge covariant manner
- $\mp\partial_{\mu}$ pulling back $x \pm n_{\mu}$ to x by shift operation

Group and Algebraic aspect of the formulation

Thus, it is natural to try to separate these two contributions :

$$e^{\mp\nabla_{\pm\mu}} \quad = \quad \underline{(\mathcal{U}_{\pm\mu})_{x,x\mp n_\mu}}\, \underline{e^{\mp\partial_\mu}}$$

where

- $$(\mathcal{U}_{+\mu})_{x,x-n_\mu} \equiv e^{-\nabla_{+\mu}}\, e^{+\partial_\mu} = e^{-\partial_\mu + i\Gamma^+_\mu}\, e^{+\partial_\mu}$$
$$= e^{+i\Gamma^+_\mu - \frac{i}{2}[\partial_\mu, \Gamma^+_\mu] + \cdots}, \qquad (\mu : \text{no sum}) \quad ,$$

- $$(\mathcal{U}_{-\mu})_{x,x+n_\mu} = e^{+\nabla_{-\mu}}\, e^{-\partial_\mu} = e^{+\partial_\mu - i\Gamma^-_\mu}\, e^{-\partial_\mu}$$
$$= e^{-i\Gamma^-_\mu - \frac{i}{2}[\partial_\mu, \Gamma^-_\mu] + \cdots}, \qquad (\mu : \text{no sum}) \quad .$$

with Link gauge covariance

$$\boxed{(\mathcal{U}_{\pm\mu})_{x,x\mp n_\mu} \;\to\; \mathcal{G}^{-1}(x)\, (\mathcal{U}_{\pm\mu})_{x,x\mp n_\mu}\, \mathcal{G}(x \mp n_\mu)}$$

Group and Algebraic aspect of the formulation

In terms of $(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x}$, the exponentiated algebra ☆ can be re-expressed as:

$$
\begin{aligned}
\{\nabla_A(x),\nabla_B(x)\} &= -f_{AB}^{\mu}\,(\mathcal{U}_{+\mu})_{x,x-n_\mu}e^{-\partial_\mu},\\
&= -f_{AB}^{\mu}\,\underline{e^{-\partial_\mu}}\,(\mathcal{U}_{+\mu})_{x+n_\mu,x},\\
\{\nabla_A(x),\nabla_B(x)\} &= +f_{AB}^{\mu}\,(\mathcal{U}_{-\mu})_{x,x+n_\mu}e^{+\partial_\mu},\\
&= +f_{AB}^{\mu}\,\underline{e^{+\partial_\mu}}\,(\mathcal{U}_{-\mu})_{x-n_\mu,x}
\end{aligned}
$$

⇩ Try to redistribute $e^{\mp\partial_\mu}$ to left hand sides

Group and Algebraic aspect of the formulation

$$\begin{aligned}
&\underline{e^{+a_A\cdot\partial}}\nabla_A(x+a_B)\,\underline{e^{+a_B\cdot\partial}}\nabla_B(x)\\
&\quad+\underline{e^{+a_B\cdot\partial}}\nabla_B(x+a_A)\,\underline{e^{+a_A\cdot\partial}}\nabla_A(x) \;=\; -f^{\mu}_{AB}\,(\mathcal{U}_{+\mu})_{x+n_\mu,x},
\end{aligned}$$

in a case where $\underline{a_A+a_B=+n_\mu}$ is satisfied, or

$$\begin{aligned}
&\underline{e^{+a_A\cdot\partial}}\nabla_A(x+a_B)\,\underline{e^{+a_B\cdot\partial}}\nabla_B(x)\\
&\quad+\underline{e^{+a_B\cdot\partial}}\nabla_B(x+a_A)\,\underline{e^{+a_A\cdot\partial}}\nabla_A(x) \;=\; +f^{\mu}_{AB}\,(\mathcal{U}_{-\mu})_{x-n_\mu,x},
\end{aligned}$$

in a case where $\underline{a_A+a_B=-n_\mu}$ is satisfied.

⇩

express $e^{+a_A\cdot\partial}\nabla_A(x)$ as $(\nabla_A)_{x+a_A,x}$,
$e^{+a_B\cdot\partial}\nabla_B(x)$ as $(\nabla_B)_{x+a_B,x}$,

Group and Algebraic aspect of the formulation

$$\{\nabla_A, \nabla_B\}_{x+a_A+a_B,x} = -f^{\mu}_{AB}\,(\mathcal{U}_{+\mu})_{x+n_\mu,x}, \quad \text{for} \quad a_A + a_B = +n_\mu,$$

$$\{\nabla_A, \nabla_B\}_{x+a_A+a_B,x} = +f^{\mu}_{AB}\,(\mathcal{U}_{-\mu})_{x-n_\mu,x}, \quad \text{for} \quad a_A + a_B = -n_\mu,$$

which are nothing but generic expressions of
N=D=4 SYM constraints on a Lattice:

$$\begin{aligned}
\{\nabla, \nabla_\mu\}_{x+a+a_\mu,x} &= +i(\mathcal{U}_{+\mu})_{x+n_\mu,x}, & a + a_\mu &= +n_\mu \\
\{\nabla_{\rho\sigma}, \nabla_\mu\}_{x+a_{\rho\sigma}+a_\mu,x} &= +i\delta_{\rho\sigma\mu\nu}(\mathcal{U}_{-\nu})_{x-n_\nu,x}, & a_{\rho\sigma} + a_\mu &= -|\delta_{\rho\sigma\mu\nu}|n_\nu \\
\{\nabla_{\rho\sigma}, \tilde{\nabla}_\mu\}_{x+a_{\rho\sigma}+\tilde{a}_\mu,x} &= +i\epsilon_{\rho\sigma\mu\nu}(\mathcal{U}_{+\nu})_{x+n_\nu,x}, & a_{\rho\sigma} + \tilde{a}_\mu &= +|\epsilon_{\rho\sigma\mu\nu}|n_\nu \\
\{\tilde{\nabla}, \tilde{\nabla}_\mu\}_{x+\tilde{a}+\tilde{a}_\mu,x} &= -i(\mathcal{U}_{-\mu})_{x-n_\mu,x}, & \tilde{a} + \tilde{a}_\mu &= -n_\mu
\end{aligned}$$

NOTE: Once expressed by link(anti)commutators, the formulation can be described by discrete lattice cites

More on Group and Algebraic aspect of the formulation

Before exponentiating the bosonic covariant derivative, the algebra:

$$\{\nabla_A, \nabla_B\} = f^{\mu}_{AB} \nabla_{\pm\mu}$$

is invariant under scale transformation D:

$$[\mathbf{D}, \nabla_A] = \frac{1}{2}\nabla_A, \quad [\mathbf{D}, \nabla_B] = \frac{1}{2}\nabla_B, \quad [\mathbf{D}, \nabla_{\pm\mu}] = \nabla_{\pm\mu},$$

After the exponentiation, the above D is no longer a symmetry.

$$\{\nabla_A, \nabla_B\} = -f^{\mu}_{AB}\, e^{-\nabla_{+\mu}},$$
$$\{\nabla_A, \nabla_B\} = +f^{\mu}_{AB}\, e^{+\nabla_{-\mu}},$$

$$[\mathbf{D}, \nabla_A] = \frac{1}{2}\nabla_A, \quad [\mathbf{D}, \nabla_B] = \frac{1}{2}\nabla_B, \quad [\mathbf{D}, \nabla_{\pm\mu}] = \nabla_{\pm\mu},$$

However, if we introduce component-wise eigenvalues d_A, d_B :

$$[\mathbf{D}, \nabla_A] = d_A \nabla_A, \quad [\mathbf{D}, \nabla_B] = d_B \nabla_B,$$

and consider a finite trans. of D: $e^{\mathbf{D}}(\cdots)e^{-\mathbf{D}}$

More on Group and Algebraic aspect of the formulation

We then obtain Weyl - 'tHooft type algebra

which have special solutions: (under gauge fields switched off)

$$
\begin{aligned}
e^{d_A+d_B}\, e^{-\nabla_{+\mu}} &= e^{\mathbf{D}}\, e^{-\nabla_{+\mu}}\, e^{-\mathbf{D}}, \\
e^{d_A+d_B}\, e^{+\nabla_{-\mu}} &= e^{\mathbf{D}}\, e^{+\nabla_{-\mu}}\, e^{-\mathbf{D}},
\end{aligned}
$$

$$
\begin{aligned}
[\partial_\mu, \mathbf{D}] &= d_A + d_B, \\
[\partial_\mu, \mathbf{D}] &= -(d_A + d_B)
\end{aligned}
$$

If $\boldsymbol{d_A, d_B}$ satisfy Lattice Leibniz rule

$$
\begin{aligned}
(d_A + d_B)_\nu &= +(n_\mu)_\nu, \quad for \quad e^{-\nabla_{+\mu}}, \\
(d_A + d_B)_\nu &= -(n_\mu)_\nu, \quad for \quad e^{+\nabla_{-\mu}},
\end{aligned}
$$

$$
[\boldsymbol{\partial_\mu, x_\nu}] = \boldsymbol{(n_\nu)_\mu = \delta_{\nu\mu}}
$$

D serves as a position operator $\boldsymbol{x_\nu}$.

It looks as if the scale operator D splits to each direction of position operator $\boldsymbol{x_\nu}$.

More on Group and Algebraic aspect of the formulation

To emphasize once again: after exponentiating the bosonic covariant derivative, the scale transformation D is no longer a symmetry of ★ .

$$
★ \quad
\begin{aligned}
\{\nabla_A, \nabla_B\} &= -f^{\mu}_{AB}\, e^{-\nabla_{+\mu}}, \\
\{\nabla_A, \nabla_B\} &= +f^{\mu}_{AB}\, e^{+\nabla_{-\mu}},
\end{aligned}
$$

$$
[\mathbf{D}, \nabla_A] = \frac{1}{2}\nabla_A, \quad \cancel{[\mathbf{D}, \nabla_B] = \frac{1}{2}\nabla_B}, \quad [\mathbf{D}, \nabla_{\pm\mu}] = \nabla_{\pm\mu},
$$

But, the position operator $\boldsymbol{x_\nu}$ satisfying:

$$
[x_\nu, \nabla_A] = (a_A)_\nu \nabla_A, \quad [x_\nu, \nabla_B] = (a_B)_\nu \nabla_B,
$$

$$
\begin{aligned}
a_A + a_B &= +n_\mu, \\
a_A + a_B &= -n_\mu,
\end{aligned}
$$

is a symmetry of ★ , provided the Lattice Leibniz rule conditions and Weyl - ‘tHooft algebra are satisfied.

$$
\begin{aligned}
e^{x_\nu}\, e^{-\nabla_{+\mu}} &= e^{\delta_{\nu\mu}}\, e^{-\nabla_{+\mu}}\, e^{x_\nu}, \\
e^{x_\nu}\, e^{+\nabla_{-\mu}} &= e^{-\delta_{\nu\mu}}\, e^{+\nabla_{-\mu}}\, e^{x_\nu},
\end{aligned}
$$

More on Group and Algebraic aspect of the formulation

COMMENT 1: the non-commutativity relations:

$$[x_\nu, \nabla_A] = (a_A)_\nu \nabla_A, \quad [x_\nu, \nabla_B] = (a_B)_\nu \nabla_B,$$

Formulations with
Link (anti)commutators
which we have constructed

- imply that fermionic Link covariant derivative $(\nabla_A)_{x+a_A,x} = e^{+a_A\cdot\partial}\nabla_A(x)$

does NOT have non-commutativity in the sense of $[x_\nu, e^{+a_A\cdot\partial}\nabla_A(x)] = 0$.

- whereas also imply non-commutativity among $\boldsymbol{x_\nu}$ and (non-link) $\boldsymbol{\theta_A}$:

$$[x_\nu, \frac{\partial}{\partial\theta_A}] = (a_A)_\nu \frac{\partial}{\partial\theta_A}, \qquad [x_\nu, \theta_A] = -(a_A)_\nu \theta_A.$$

Formulations with non(anti)commutative superspace

More on Group and Algebraic aspect of the formulation

COMMENT 2: Remind that:

$$[x_\nu, \frac{\partial}{\partial\theta_A}] = (a_A)_\nu \frac{\partial}{\partial\theta_A}, \qquad [x_\nu, \theta_A] = -(a_A)_\nu \theta_A.$$

☆
$$\{\nabla_A, \nabla_B\} = -f^{\mu}_{AB}\, e^{-\nabla_{+\mu}},$$
$$\{\nabla_A, \nabla_B\} = +f^{\mu}_{AB}\, e^{+\nabla_{-\mu}},$$

stems from the exponentiation of $\boldsymbol{\nabla}_{\pm\mu}$ in the r.h.s. of ☆ .

If we start from the ordinary anti-commutative relation: $\{\xi_A, \xi_B\} = 0$

we obtain

$$(\xi_A)_{x-a_A,x} \equiv e^{-a_A\cdot\partial}\xi_A, \qquad \text{with} \quad [x_\nu, \xi_A] = 0$$

which satisfies

- $\{\boldsymbol{\nabla}_A, \boldsymbol{\xi}_B\}_{x+a_A-a_B,x} = 0$
- $(\boldsymbol{\xi}_A)_{x+a_A,x} \rightarrow G^{-1}_{x+a_A}\, (\boldsymbol{\xi}_A)_{x+a_A,x}\, G_x = (\boldsymbol{\xi}_A)_{x+a_A,x}$

Thus, $(\boldsymbol{\xi}_A)_{x+a_A,x}$ can serve as Grassmann link parameters ensuring SUSY inv. of the Action.

Short Summary of Group and Algebraic aspects

- Promotion of super-covariant derivatives $\nabla_{\pm\mu}$ to Group element provides Link (anti)commutator formulation and $(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x}$.

  ⇨ Microscopic understanding of Lattice super Yang-Mills.

- After the promotion, it looks as if the scale operator D splits to each direction of position operator x_ν .

⇨ Provides algebraic understanding of Grassmann link parameter ξ_A

⇨ Expectation of Large Symmetry including spacetime structure behind the formulation. As large as N=D=4 superconformal ?

Part 2: Non(anti)commutative superspace

K.N., arXiv:2502.16410 [hep-th]

Consider $\boldsymbol{Q_A}$, $\boldsymbol{\theta_A}$, $\boldsymbol{\xi_A}$ with non(anti)commutativities (NAC):

$$
\begin{aligned}
\{Q_A, Q_B\} &= P_{AB}, \quad Q_A^2 = Q_B^2 = 0, \\
\{\theta_A, \theta_B\} &= a_{AB}, \quad \theta_A^2 = \theta_B^2 = 0. \\
\{\xi_A, \xi_B\} &= c_{AB}, \quad \xi_A^2 = \xi_B^2 = 0, \\
\{\xi_A, \theta_B\} = \{\xi_B, \theta_A\} &= b_{AB}, \quad \{\xi_A, \theta_A\} = \{\xi_B, \theta_B\} = 0
\end{aligned}
$$

NAC in vector sector

and define:

$$
X \equiv \xi_A Q_A + \xi_B Q_B, \qquad Y \equiv \theta_A Q_A + \theta_B Q_B, \qquad (A, B : \text{no sum})
$$

we then have:

$$
\begin{aligned}
[X, [X, Y]] &= \gamma Y + \beta X, \\
[Y, [Y, X]] &= \alpha X + \beta Y,
\end{aligned}
$$

$$
\begin{aligned}
\alpha &\equiv -a_{AB} P_{AB}, \\
\beta &\equiv +b_{AB} P_{AB}, \\
\gamma &\equiv -c_{AB} P_{AB}.
\end{aligned}
$$

Non(anti)commutative superspace

which induces Infinite dimensional Lie Algebra:

$$\begin{aligned} X_{l,m,n} &\equiv \alpha^l \beta^m \gamma^n X, \qquad l,m,n = 0,1,2,\ldots \\ Y_{l,m,n} &\equiv \alpha^l \beta^m \gamma^n Y, \qquad l,m,n = 0,1,2,\ldots \\ Z_{l,m,n} &\equiv \alpha^l \beta^m \gamma^n [X,Y], \qquad l,m,n = 0,1,2,\ldots, \\ [X_{l,m,n}, Y_{l',m',n'}] &= Z_{l+l',m+m',n+n'} \\ [X_{l,m,n}, Z_{l',m',n'}] &= X_{l+l',m+m'+1,n+n'} + Y_{l+l',m+m',n+n'+1} \\ [Y_{l,m,n}, Z_{l',m',n'}] &= -X_{l+l'+1,m+m',n+n'} - Y_{l+l',m+m'+1,n+n'}, \end{aligned}$$

$$\begin{aligned} \alpha &\equiv -a_{AB} P_{AB}, \\ \beta &\equiv +b_{AB} P_{AB}, \\ \gamma &\equiv -c_{AB} P_{AB}. \end{aligned}$$

$$X \equiv \xi_A Q_A + \xi_B Q_B, \qquad Y \equiv \theta_A Q_A + \theta_B Q_B, \qquad (A, B : \text{no sum})$$

Non(anti)commutative superspace

Multiplication of superspace group elements becomes highly non-linear:

$$e^X e^Y \quad = \quad \exp\left[F(\alpha,\beta,\gamma)\Big(G(\alpha)X + G(\gamma)Y + \frac{1}{2}[X,Y]\Big)\right],$$ closed form BCH with

$$F(\alpha,\beta,\gamma) =$$

$$\frac{\log\left(\operatorname{ch}\sqrt{\tfrac{1}{4}\alpha}\operatorname{ch}\sqrt{\tfrac{1}{4}\gamma} - \tfrac{\beta}{\sqrt{\alpha\gamma}}\operatorname{sh}\sqrt{\tfrac{1}{4}\alpha}\operatorname{sh}\sqrt{\tfrac{1}{4}\gamma} + \sqrt{\left(\operatorname{ch}\sqrt{\tfrac{1}{4}\alpha}\operatorname{ch}\sqrt{\tfrac{1}{4}\gamma} - \tfrac{\beta}{\sqrt{\alpha\gamma}}\operatorname{sh}\sqrt{\tfrac{1}{4}\alpha}\operatorname{sh}\sqrt{\tfrac{1}{4}\gamma}\right)^2 - 1}\right)}{\frac{\sqrt{\frac{1}{4}\alpha}\sqrt{\frac{1}{4}\gamma}}{\operatorname{sh}\sqrt{\frac{1}{4}\alpha}\ \operatorname{sh}\sqrt{\frac{1}{4}\gamma}}\sqrt{\left(\operatorname{ch}\sqrt{\tfrac{1}{4}\alpha}\operatorname{ch}\sqrt{\tfrac{1}{4}\gamma} - \tfrac{\beta}{\sqrt{\alpha\gamma}}\operatorname{sh}\sqrt{\tfrac{1}{4}\alpha}\operatorname{sh}\sqrt{\tfrac{1}{4}\gamma}\right)^2 - 1}},$$

$$G(\alpha) = \frac{\sqrt{\frac{1}{4}\alpha}\ \operatorname{ch}\sqrt{\frac{1}{4}\alpha}}{\operatorname{sh}\sqrt{\frac{1}{4}\alpha}}, \qquad G(\gamma) = \frac{\sqrt{\frac{1}{4}\gamma}\ \operatorname{ch}\sqrt{\frac{1}{4}\gamma}}{\operatorname{sh}\sqrt{\frac{1}{4}\gamma}},$$

$$\begin{aligned} \alpha &\equiv -a_{AB}P_{AB}, \\ \beta &\equiv +b_{AB}P_{AB}, \\ \gamma &\equiv -c_{AB}P_{AB}. \end{aligned}$$

Non(anti)commutative superspace

However, if we consider a problem setting corresponding to Lattice SUSY:

$$
\begin{aligned}
\{Q_A, Q_B\} &= \frac{+i}{n_{AB}} e^{-in_{AB}P_{AB}}, & \{\theta_A, \theta_B\} &= -i\, a_{AB}\, e^{+in_{AB}P_{AB}}, \\
\{Q_A, Q_B\} &= \frac{-i}{n_{AB}} e^{+in_{AB}P_{AB}}, & \{\theta_A, \theta_B\} &= +i\, a_{AB}\, e^{-in_{AB}P_{AB}}, \\
& & \{others\} &= 0,
\end{aligned}
$$

compatible with

$$
\begin{aligned}
[x, \theta_A] &= -a_A \theta_A \\
[x, \theta_B] &= -a_B \theta_B.
\end{aligned}
$$

$\boldsymbol{n_{AB}}$: lat. const.
$\boldsymbol{a_{AB}}$: mag. of NAC

the non-linear terms turn to be governed by a ratio factor $\boldsymbol{r_a = a_{AB}/n_{AB}}$.

In a particular example, if we consider N=D=4 case:

$$
\begin{aligned}
X &= \xi Q + \xi_\mu Q_\mu + \frac{1}{2}\xi_{\mu\nu} Q_{\mu\nu} + \tilde{\xi}_\mu \tilde{Q}_\mu + \tilde{\xi}\tilde{Q}, \\
Y &= \theta Q + \theta_\mu Q_\mu + \frac{1}{2}\theta_{\mu\nu} Q_{\mu\nu} + \tilde{\theta}_\mu \tilde{Q}_\mu + \tilde{\theta}\tilde{Q}, \quad (\mu, \nu = 1 \sim 4\text{: summed up})
\end{aligned}
$$

and take: $\boldsymbol{\tan\sqrt{r_a} = \sqrt{r_a}}$, then

$$
e^X e^Y = \exp\left[X + Y + \frac{1}{2}[X, Y]\right]
$$

is EXACT.

Summary & Discussions

- N=D=4 Twisted SYM on a Lattice is constructed via Link formulation.
  - Lattice SUSY Algebra and SUSY inv. for All Supercharges
- Group and Algebraic aspects of the formulation have been revealed, which provides microscopic understanding of Lattice SUSY.
  - Gauge Link variables $(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x}$ naturally obtained by exp SUSY algebra
  - Position operators x_μ appeared as symm. generators of exp SUSY algebra

  ⇨ Realization of "Algebra defines spacetime."
- Non(anti)commutative superspace provides positive implications.
- Further studies are necessary for Quantum and Numerical aspects.
- Expectation of Larger Symmetry behind the formulation.

Thank you so much for your attention