Matrix regularization of Lie-Poisson algebra

Lie-Poisson 代数の行列正則化

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0. Back ground

1. Intro, Motivation

1-1 Semisimple Lie alg is solution of IKKT.

1-2 Fuzzy S^2

2. Matrix regularization (MR) for $\mathbb{C}[x]$

3. MR for $\mathbb{C}[x]/I(C)$

$$\underbrace{\mathbb{C}[x]/I(C) \xrightarrow{\xi_{\nu/I}} U_{\hbar}[\hbar]/I(C)}_{3-1} \; \underbrace{\xrightarrow{\rho} gl(V^{\mu}) \xrightarrow{R} gl(V^{\mu})}_{3-2}$$

4. Reducible representation

5. Summary

0. Back ground

String theory : candidate of Quantum gravity

↓ non-perturbative

- Matrix model. (IKKT matrix model)

  [Ishibashi - Kawai - Kitazawa - Tsuchiya '97)

Solution of E.O.M. is given by "Matrix".

$\leadsto$ N.C. geometry with some matrix rep.

We have to find some connection

between "Matrix geometry" & commutative geo.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ||

A method is "Matrix regularization" Poisson alg.

(Hoppe, Madore)

↑

generalization of Berezin - Toeplitz Quantization

0-2

▷ Matrix regularization (Hoppe 89, Madore 92)

$N_1, N_2, N_3, \cdots$: increasing seq of positive integers

$\hbar(N)$: $|\lim_{N\to\infty} N\hbar(N)| < \infty$ converge.

(M,ω): Symplectic mfd.

$T_k : C^\infty(M) \longrightarrow \{N_k \times N_k \text{ Hermitian}\} \subset Mat_{N_k}(\mathbb{C})$

s.t. • $\lim_{k\to\infty} \| T_k(f) \| < \infty$ ✓

• $\lim_{k\to\infty} \| T_k(fg) - T_k(f)T_k(g) \| = 0$ ✓

• $\lim_{k\to\infty} \| \frac{1}{i\hbar(N)} [T_k(f), T_k(g)] - T_k(\{f,g\}) \| = 0$ ✓

• $\lim_{k\to\infty} 2\pi\hbar(N_k) \, \mathrm{Tr}(T_k(f)) = \int_M f\omega$

Except for B-T quantization, only a few examples are known.

Def) Quantization

A : Poisson alg. $(A, \cdot, \{ , \})$

$T \subset M$: R-module in R-alg M (R is $\mathbb{C}$ or $\mathbb{C}[\hbar]$)

$g : A \to T$ linear, with $\hbar(g) \in \mathbb{C}$

for $\forall f, g \in A$,

$$[g(f), g(g)] = i\hbar\, g(\{f, g\}) + \hbar^{1+\varepsilon}$$

1. Intro. Motivation

1-1 Semisimple Lie alg is solution of Mass-def. IKKT

▷ Matrix model (M.M.) of $\mathfrak{g}$-solution

$\mathfrak{g}$: semi simple Lie alg

basis :

$[X_i, X_j] = \hbar f_{ij}^{k} X_k \quad , \quad \mathrm{tr}\, X^i X_j = c\, \delta^i_j$

$g_{ij} := - f_{il}^{m} f_{jm}^{l}$: Killing metric.

(Non-degenerate for semisimple)

$$\boxed{S = -\frac{1}{4} g^{ij} g^{kl}\, \mathrm{tr}\, [X_i, X_k][X_j, X_l] + \frac{\hbar^2}{2} g^{ij}\, \mathrm{tr}\, X_i X_j}$$

$\rightsquigarrow$ E.O.M.

$[X^i, [X_i, X_j]] = -\hbar^2 X_j$

Prop. The above basis is a solution.

Proof).

$$[X^i,[X_i,X_j]] = \hbar f_{ij}^{\ \ k} g^{il}[X_l,X_k]$$
$$= \hbar^2 f_{ij}^{\ \ k} g^{il} f_{lk}^{\ \ m} X_m$$
$$= \hbar^2 f_{ij}^{\ \ k} g^{il} g^{mn} f_{lkn} X_m$$

From $\mathrm{Tr}\, X^i X_j = c\,\delta^i{}_j$, f_{ijk} is fully anti-sym

$$= \hbar^2 f_{ji}^{\ \ k} f_{nkl} g^{il} g^{mn} X_m$$
$$= \hbar^2 f_{ji}^{\ \ k} f_{nk}^{\ \ \ i} g^{mn} X_m$$
$$= -\hbar^2 g_{jn} g^{mn} X_m = -\hbar^2 X_j$$

//

SO(n, d-n) type metric

$$(\eta) = \mathrm{diag}(\underbrace{1, 1, \cdots, 1}_{n}, \underbrace{-1, \cdots, -1}_{d-n})$$

$$O^{-1}(g^{\mu\nu})O = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_n & & & \\ & & & -\lambda_{n+1} & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda_d \end{pmatrix} = (\lambda_\tau \eta^{\tau\tau})$$

$\Downarrow$ Change of variables

$$\boxed{X'_\tau := \sum_\nu \sqrt{\lambda_\tau}\, O_{\nu\tau} X_\nu}$$

$$S = S_{IKKT} = \mathrm{tr}\left(-\frac{1}{4}[X_\mu, X_\nu]^2 + \underbrace{\hbar^2 \frac{1}{2} X^\mu X_\mu}_{\text{mass-deformed}}\right)$$

Thm. [Arnlind-Hoppe '09]

mass-def- IKKT M.M. with $\eta = (\underbrace{+1, \cdots, +1}_{n}, \underbrace{-1, \cdots, -1}_{d-n})$

has solution

$$X'_\mu = \sum_\nu \sqrt{\lambda_\tau}\, O_{\nu\tau} X_\nu .$$

where X_ν is basis of $\mathfrak{g}$ whose Killing (n, d-n) type metric

semisimple

$\mathfrak{g}$ is solution of IKKT.

$\downarrow$

$\boxed{\text{Q. What is classical physics of } \mathfrak{g}\text{?}}$

"Poisson-Lie algebra"

1-2. Fuzzy S^2, $\mathbb{R}^3$ (Madre, Hoppe)

~ As a Hint of such classical physics ~

Consider $su(2)$ case

$(x_1, x_2, x_3) \in \mathbb{R}^3$ with $su(2)$ Poisson str.

$$\{x_a, x_b\} = i\epsilon^{abc} x_c$$ ← Kirillov-Kostant $\mathfrak{g}(\mathfrak{g}^*) \to$ Poisson alg.

$A_{su(2)} = (\mathbb{C}[x], \cdot, \{,\})$: Poisson-Lie alg.

$f \in A_{su(2)}$: Polynomial on $\mathbb{R}^3$

$$f = f_0 + f_a x_a + \frac{1}{2} f_{ab} x_a x_b + \cdots$$

$f_{a_1 \cdots a_n}$: completely sym.

V^k : k-dim Vect. sp.

$\rho_k : A_{su(2)} \longrightarrow gl(V^k)$: linear

V^k : k-dim Vect. sp.

$\mathcal{g}_k$: $A_{su(2)} \longrightarrow gl(V^k)$: linear

$$x^A = x_{a_1} \cdots x_{a_m} \mapsto \mathcal{g}_k(x^A) = \begin{cases} \dfrac{\hbar^m}{m!} \displaystyle\sum_{\sigma \in Sym(m)} J_{a_{\sigma(1)}} \cdots J_{a_{\sigma(m)}} & (m < k) \\ 0 & (m \geq k) \end{cases}$$

where J_i $(i = 1 \sim 3)$ is $su(2)$ basis

$$[J_i, J_j] = i\epsilon^{ijk} J_k \qquad (\text{Spin } 2s+1 = k \text{ rep.})$$

$$\boxed{\mathcal{g}_k(x_i) = \hbar J_i}$$

Not S^2 yet (Quantization of $\mathbb{R}^3$)

◎ $su(2)$ has a quadratic Casimir

Def). Casimir Polynomial

$\mathfrak{g}$: d-dim Lie alg.

$A_{\mathfrak{g}} := (\mathbb{C}[x], \cdot, \{\ ,\ \})$

$x = (x_1, \cdots, x_d)$

Kirillov-Kostant

$[X_i, X_j] = f_{ij}^k X_k$

$\{x_i, x_j\} = f_{ij}^k x_k$

Lie-Poisson alg.

$f(x)$ is a Casimir poly. $\overset{\text{def}}{\Longleftrightarrow}$ $\{x_i, f(x)\} = 0$

$i = 1, 2, \cdots, d$

For $A_{su(2)}$ $f^c = \delta^{ab} x_a x_b$ is the Casimir

$$Q_k(f^c(x)) = \hbar^2 \delta^{ab} J_a J_b \quad : \text{Casimir op.}$$

$$= \hbar^2 \frac{1}{4}(k^2 - 1)\mathbb{1}_k$$

$\rightsquigarrow$ If we fix $\hbar(Q_k) = \sqrt{\dfrac{4\lambda^2}{k^2-1}}$

then $\displaystyle\lim_{k\to\infty} \delta^{ab} Q_k(x_a) Q_k(x_b) = \underline{\lambda^2}$

fixed

$[Q_k(x_a), Q_k(x_b)] = \hbar\, Q_k(\{x_a, x_b\}) \longrightarrow 0$ commutative limit

$(k \to \infty)\, \hbar \to 0$

$\delta^{ab} x_a x_b = \lambda^2 \; : \; S^2$

su(2) rep is quantization of S^2

◎ Polynomial on S^2

i) a formulation : spherical harmonics

$\uparrow$

I don't know how to generalize this.

ii) function on S^2

$\mathbb{C}[x]$: functions on $\mathbb{R}^3$

$$I(C) := \{ g(x)(\underbrace{\delta^{ab}x_a x_b - \lambda^2}_{\text{Eq of } S^2}) \mid g(x) \in \mathbb{C}[x] \}$$

$I(C)$ is an ideal.

i.e.) $I(C) \subset \mathbb{C}[x]$, module

$\forall f \in I(C), \forall g(x) \in \mathbb{C}[x] \Rightarrow f(x)g(x) \in I(C)$

$$I(C) := \{\, g(x)(\underline{\delta^{ab}x_a x_b - \lambda^2}) \mid g(x) \in \mathbb{C}[x] \,\}$$

$$\Downarrow$$

$\mathbb{C}[x]/I(C)$: functions on S^2

$\Downarrow$

$$[f] = \{ f + h \mid f \in \mathbb{C}[x],\ h \in I(C) \}$$

$$\left.\begin{aligned} [f]+[g] &:= [f+g] \\ [f]\cdot[g] &:= [f\cdot g] \\ \{[f],[g]\} &:= [\{f,g\}] \end{aligned}\right\} \text{well-defined}$$

Variety S^2 is given by $\mathbb{C}[x]/I(C)$

$\rightarrow$ generalize this picture.

2 M.R. for $\mathbb{C}[x]$ with $\mathfrak{g}$

$\{e_1, \cdots, e_d \mid$ basis of $\mathfrak{g}$ $\quad [e_i, e_j] = f_{ij}^k e_k \}$

$\rho^\mu : \mathfrak{g} \to \mathfrak{gl}(V^\mu)$: irreducible rep.

$e_i^{(\mu)} := \underline{\hbar}\, \rho^\mu(e_i)$, $\quad [e_i^{(\mu)}, e_j^{(\mu)}] = \hbar\, f_{ij}^k\, e_k^{(\mu)}$

T^μ = alg generated by $\{e_1^{(\mu)}, \cdots, e_d^{(\mu)}\}$.

Notation

$$\boxed{e^{(\mu)}_{I_i^m} := e^{(\mu)}_{(i_1, \cdots, i_m)} = \frac{1}{m!} \sum_{\sigma \in \mathrm{Sym}(m)} e^{(\mu)}_{i_{\sigma(1)}} \cdots e^{(\mu)}_{i_{\sigma(m)}}}$$

$I_j^k = \{\{ j_1, \cdots, j_k \}\}$, $\quad I_i^k \sqcup I_j^l = \{\{ i_1, \cdots, i_k, j_1, \cdots, j_l \}\}$

We can make a basis of T_μ by $E_{I_i^m} := e^{(\mu)}_{I_i^m}$.

D) "Weak" M.R.

Def). $\mathcal{g}_\mu : A_q \to T_\mu$ linear s.t. $:= (\mathbb{C}[x], \{,\})$

for $x^I = x_{i_1} \cdots x_{i_m}$

$$\mathcal{g}_\mu(x^I) = \begin{cases} e^{(\mu)}_{(i_1 \cdots i_m)} = \frac{1}{m!} \sum_{\sigma \in S_m} e^{(\mu)}_{i_{\sigma(1)}} \cdots e^{(\mu)}_{i_{\sigma(m)}} & (m \le n_\mu) \\ 0 & (m > n_\mu) \end{cases}$$

> Def. $\mathcal{g}_k$: Poisson alg $\to gl(V_k)$ is Quantization
>
> $$[\mathcal{g}_k(f), \mathcal{g}_k(g)] = \hbar\, \mathcal{g}_k(\{f,g\}) + \hbar^2 (\Sigma a_I E_I),$$
>
> with $\hbar \to 0$ when $\dim V_k \to \infty$. We say $\mathcal{g}_k$ is "Weak M.R."

We can prove the above $\mathcal{g}_\mu$ is "Weak M.R."

3. M.R. for $A_g / I(C)$

3-0. Preparation

▷ Casimir (k-th degree) C_k

For Semisimple Lie alg

There exists $\{V^\mu : \text{Ir. rep. sp. of } \mathfrak{g}\}$ s.t.

$$\dim V^\mu \xrightarrow[\mu\to\infty]{} \infty \quad \cap \quad |C_k(V^\mu)| \xrightarrow[\mu\to\infty]{} \infty$$

$$C_k(V^\mu) = \Lambda^k \, \mathrm{Id}(V^\mu)$$

Assume

$\mathfrak{g}$ has series of Ir. rep. $\{V^\mu\}$ s.t.

k-th degree Casimir

$$C_{k\,i} = \Lambda^k_i(V^\mu)\, \mathrm{Id}_\mu , \quad \lim_{\dim V^\mu \to \infty} |\Lambda^k_i(V^\mu)| = \infty$$

<u>Prop.</u> $C_i^k(x) \in A_{\mathfrak{g}}$ is Casimir poly of deg k

i.e. $\{x_i, C_i^k(x)\} = 0 \quad (i=1\sim d)$

$\Rightarrow \quad \mathcal{E}_\mu(C_i^k(x))$ is Casimir op.

We denote $\quad C_i^k(e^\mu) := \mathcal{E}_\mu(C_i^k(x)) \sim \hbar^k$

We fix $\hbar(\mu)$ by $|C_i^k(e^\mu)| = |\hbar^k \Lambda_i^k| = \text{const}.$

$$\boxed{C_i^k(e^\mu) = \lambda_i^k}$$

▷ Gröbner basis

We introduce a monomial order.

$$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n},\quad x^\beta = x_1^{\beta_1}\cdots x_n^{\beta_n}$$

Def). graded lexicographic ordering

$$\alpha<\beta \overset{\text{def}}{\iff} \begin{cases} \deg x^\alpha = \alpha_1+\cdots+\alpha_n < \deg x^\beta = \beta_1+\cdots+\beta_n \\ \alpha_1=\beta_1,\cdots,\alpha_{i-1}=\beta_{i-1},\ \alpha_i<\beta_i \quad , \quad \deg x^\alpha = \deg x^\beta \end{cases}$$

"multi degree"

ex) $\mathrm{multdeg}(y^3) > \mathrm{multdeg}(x^2)$

$\mathrm{multdeg}(x^2y) > \mathrm{multdeg}(xy^2)$

~ Why Gröbner basis? ~

Consider $\mathbb{C}[x,y]$ &

I : the ideal generated by $\{xy-1,\ y^2-1\}$.

What is the $\underbrace{\text{remainder}}_{r_f}$ of $f = x^2y + xy^2 + y^2$

$f = (x+1)(y^2-1) + x(xy-1) + \underbrace{2x+1}$ r_f ?

$f = (x+y)(xy-1) + 1\cdot(y^2-1) + \underbrace{x+y+1}$ r_f ?

Not unique!

<u>Fact</u>. Fix a mono-order on $\mathbb{C}[x_1, \cdots, x_d]$.

1. Every ideal has a unique reduced <u>Gröbner basis</u> $G = \{g_1, \cdots, g_m\}$ → I's element is expanded by this basis

2. $\forall f \in \mathbb{C}[x_1, \cdots, x_d]$

$$\left.\begin{array}{l} f = h + r \\ h \in I,\ r \notin I \end{array}\right\} \text{ is uniquely determined}$$

s.t. no monomial in r is divisible by g_i.

3-1. $\delta_{1/2} : A_{\mathfrak{g}}/I(C) \longrightarrow U_{\mathfrak{g}}/I(C(X))$

$\mathfrak{g}$: d-dim Lie alg. $[e_i, e_j] = f_{ij}^k e_k$ $(i, j = 1 \sim d_c)$

$+$

k-th Casimir $C_{ki} = \Lambda_i^k(V^\mu)\, Id_\mu$, $\lim_{\dim V^\mu \to \infty} |\Lambda_i^k| = \infty$

$A_{\mathfrak{g}} := (\mathbb{C}[x], \cdot, \{\ ,\ \})$ $\{x_i, x_j\} = f_{ij}^k x_k$

Step 1

$U_{\mathfrak{g}}[\hbar]$: Univ enveloping alg.

$$\left\{ \sum_k a_{I_k}(\hbar) X^{I_k} \right\} \Big/ \left(X_i X_j - X_j X_i - [X_i, X_j]\right)_{1 \le i, j \le d}$$

$\hbar f_{ij}^k X_k$!!

$X^{I_k} = X_{i_1} \cdots X_{i_k}$ X_i: noncom variables

$\delta_U : A_{\mathfrak{g}} \longrightarrow U_{\mathfrak{g}}[\hbar]$

$\hbar$ does not appear in

$$x_{i_1} \cdots x_{i_k} \longmapsto \delta_U(x_{i_1} \cdots x_{i_k}) = \frac{1}{k!} \sum_{\sigma \in S_k} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(k)}} =: X^{I_k}$$

$\sigma_U : A_{\mathfrak{g}} \longrightarrow U_{\mathfrak{g}}[\hbar]$

$\hbar$ does not appear in

$$x_{i_1} \cdots x_{i_k} \longmapsto \sigma_U(x_{i_1} \cdots x_{i_k}) = \underbrace{\frac{1}{k!} \sum_{\sigma \in S_k} X_{i_{\sigma(1)}} \cdots X_{i_{\sigma(k)}}}_{X^I}$$

$$x_I \longmapsto X^I$$

Prop. $[\sigma_U(f), \sigma_U(g)] = \hbar\, \sigma_U(\{f, g\}) + O(\hbar^2)$

especially $[\sigma_U(x_i), \sigma_U(f)] = \hbar\, \sigma_U(\{x_i, f\})$

$\hookrightarrow$ Casimiar poly $\mapsto$ Casimir op.

Step 2 Casimir poly. (k-th)

$$\{x_i, f^C(x)\} = 0 \qquad f^C = k\text{-th homogeneous poly}$$

$A_{alg}/I(C) = \left(\mathbb{C}[x]/I(C), \cdot, \{,\}\right)$ is given by

- $I(C) = \left\{ \sum g_i(x) f_i^C \;\middle|\; g_i \in \mathbb{C}[x],\ f_i^C(x) \text{ is a Casimir} \right\}$
- $[f(x)] = \{f + h \mid h \in I\}$

$[f] + [g] := [f+g]$, $[f] \cdot [g] := [f \cdot g]$

$\{[f], [g]\} := [\{f, g\}]$

$A_{\mathfrak{g}}/I(C) = (\mathbb{C}[x]/I(C), \cdot, \{,\})$

- $I(C) = \{ \sum g_i(x) f_i^c \mid g_i \in \mathbb{C}[x],\ f_i^c(x) \text{ is a Casimir} \}$

$U_{\mathfrak{g}}[\hbar]/IC(X)$

where $IC(X) = \{ \sum a_i(x) \overbrace{(g_U(\underbrace{f^c_{(x)}}_{\text{Casimir poly.}}))}^{\text{Casimir Op.}} b_i(X) \}$

$g_{U/I} : A_{\mathfrak{g}}/I(C) \longrightarrow U_{\mathfrak{g}}[\hbar]/IC(X)$

$[f] = [r_f] \longmapsto g_{U/I}([f]) = [g_U(r_f)]$

$f = \underset{\cancel{\in}\ I(C)}{r_f} + \underset{\in\ I(C)}{h_f}$

$A_{\mathfrak{g}} \longrightarrow U_{\mathfrak{g}}$

$$\boxed{\begin{array}{l}\textbf{Thm.}\\ [g_{U/I}([f]), g_{U/I}([g])] = \hbar\, g_{U/I}(\{[f],[g]\}) + O(\hbar^2)\end{array}}$$

3-2. $\mathbb{C}[x]/I(C) \longrightarrow gl(V^{\mu})$

Let's choose special $I(C)$:

$$I(C) := \{ \textstyle\sum f_i^C(x) g_i(x) \mid f_i^C(x) = C_i^k(x) - \lambda_i^k,\ g_i(x) \in A_g \}$$

k-th deg Casimir

eigen value

$$f_i^C(X) := \mathcal{S}_U(f_i^C(x))$$

$$I(C(X)) = \{ \textstyle\sum a_i(X) f_j^C(X) b_k(X) \} \subset U_g[\hbar]$$

Def). $\rho_{U/I,\mu} : U_g[\hbar]/I(C(X)) \longrightarrow gl(V^{\mu})$

$X^I = X_{i_1} \cdots X_{i_n} \in U_g[\hbar]$ $\quad \rho_{U/I,\mu}([X^I]) = e_{i_1}^{(\mu)} \cdots e_{i_n}^{(\mu)} =: e_I^{(\mu)}$

Why well-defined?

$[f] = [g] \rightarrow f = g + h \quad (h \in I(C(X)))$ $\Rightarrow I(C(X)) \subset \operatorname{Ker} \rho_{U/I,\mu}$

$h(e^{(\mu)}) = 0 \rightarrow \rho_{U/I,\mu}([f]) = \rho_{U/I,\mu}([g])$

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$$\begin{tikzcd} \mathbb{C}[x]/I(C) \arrow[r, "g_{V/I}"] \arrow[rr, bend left, "g_\mu^{Pre}"] & U_g[\hbar]/I(C) \arrow[r, "\rho_{V/I,\mu}"] & gl(V^\mu) \arrow[r, "R_\mu"] & gl(V^\mu) \end{tikzcd}$$

Casimir $C_i(x)$ Casimir $g_\mu(C_i(x))$

Thm.

$$\boxed{\begin{gathered} g_\mu^{Pre} := \rho_{V/I,\mu} \circ g_{V/I} \text{ is Weak M.R.} \\ [g_\mu^{Pre}([f]), g_\mu^{Pre}([g])] = \hbar\, g_\mu^{Pre}(\{[f],[g]\}) \\ + \hbar^2 \times P(e^{(\omega)}) \end{gathered}}$$

▷ Projection $R_\mu : gl(V^\mu)[\hbar] \to gl(V^\mu)[\hbar]$

$$M = \sum_{0 \le k} \hbar^k M_k \qquad M_k \in gl(V^\mu)$$

$$R_\mu M = \sum_{0 \le k \le n_\mu} \hbar^k M_k .$$

$g_\mu^{Pre} : \mathbb{C}[x]/I(C) \longrightarrow gl(V^\mu)$

$$
\begin{tikzcd}
\mathbb{C}[x]/I(C) \arrow[r, "g_{V/I}"] \arrow[rrr, bend left, "g_{A/I,\mu}"] & U_{\mathfrak{g}}[\hbar]/I(C) \arrow[r, "\rho_{V/I,\mu}"] & \mathfrak{gl}(V^\mu) \arrow[r, "R_\mu"] & \mathfrak{gl}(V^\mu)
\end{tikzcd}
$$

Casimir $C_i(x)$ Casimir $g_\mu(C_i(x))$

Def). $g_{A/I,\mu} : A_{\mathfrak{g}}/I(C) \longrightarrow \mathfrak{gl}(V^\mu)$

$$g_{A/I,\mu} := R_\mu \circ g_\mu^{Pre} = R_\mu \circ \rho_{V/I,\mu} \circ g_{V/I}$$

Thm. $g_{A/I,\mu}$ is a weak M.R.

$$[g_{A/I,\mu}([f]), g_{A/I,\mu}([g])] = \hbar\, g_{A/I,\mu}(\{[f],[g]\}) + \hbar^2 P$$

Thm.

$$g_{A/I,\mu}([f]\cdot[g]) = g_{A/I,\mu}([f])\, g_{A/I,\mu}([g]) + \hbar P$$

3-3 Example

ex) $su(3)$

$[T_i, T_j] = f_{ij}^{\ k} T_k$, $\quad T_k = \frac{1}{2}\lambda_i$

$\uparrow$

Gell-Mann matrix

$\Downarrow$

$x = (x^1, \cdots, x^8)$, $\quad \{x_i, x_j\} = f_{ij}^{\ k} x_k$

$C^2(x) = \frac{1}{3} \delta^{ab} x_a x_b \longrightarrow S^7$

$C^3(x) = \frac{1}{18}\left(2\sqrt{3}\, x_8^3 - 6\sqrt{3}\, \underline{x_1^2 x_8} - \cdots \right)$

leading term

$\downarrow$

i) $I(C) = \{0\}$, $\quad$ ii) $I(C) = \langle C^2(x) - \lambda^2 \rangle$

$\rightarrow \mathbb{R}^8 \qquad\qquad \rightarrow S^7$

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iii) $I(C) = \langle C^3(x) - \lambda^3 \rangle$ case

$C^3(x) - \lambda^3 =: f^C(x) \rightsquigarrow I(C)$ is generated

- $A_{su(3)} / I(C)$ is a set of fun. on $f^C(x) = 0$.
- Gröbner $G = \{ C^3(x) - \lambda^3 \}$

$f(x) = x_1^2 x_3 x_8 + x_2$

$f = r_f + h_f$

devide by $f^C(x)$

$r_f = x_2 + \frac{1}{3} x_3 x_8^3 + \cdots$

as an example $V^\mu = \mathbb{C}^3$

$\rightsquigarrow n_\mu = 1 \quad (\underbrace{\text{Gell-Mann}}_{8} + \underbrace{\text{Id}}_{1})$

R^μ removes terms deg ≥ 2

$g_{A/I, \mu}([x_1^2 \cancel{x_3 x_8} + x_2]) = \hbar T_2$

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$$C^3 = \frac{1}{18}(2\sqrt{3}x_8^3 - 6\sqrt{3}x_1^2x_8 - 6\sqrt{3}x_2^2x_8 - 6\sqrt{3}x_3^2x_8 + 3\sqrt{3}x_4^2x_8 + 3\sqrt{3}x_5^2x_8 + 3\sqrt{3}x_6^2x_8 + 3\sqrt{3}x_7^2x_8 - 18x_2x_5x_6 + 18x_2x_4x_7 - 18x_1(x_4x_6 + x_5x_7) - 9x_3\left(x_4^2 + x_5^2 - x_6^2 - x_7^2\right)).$$

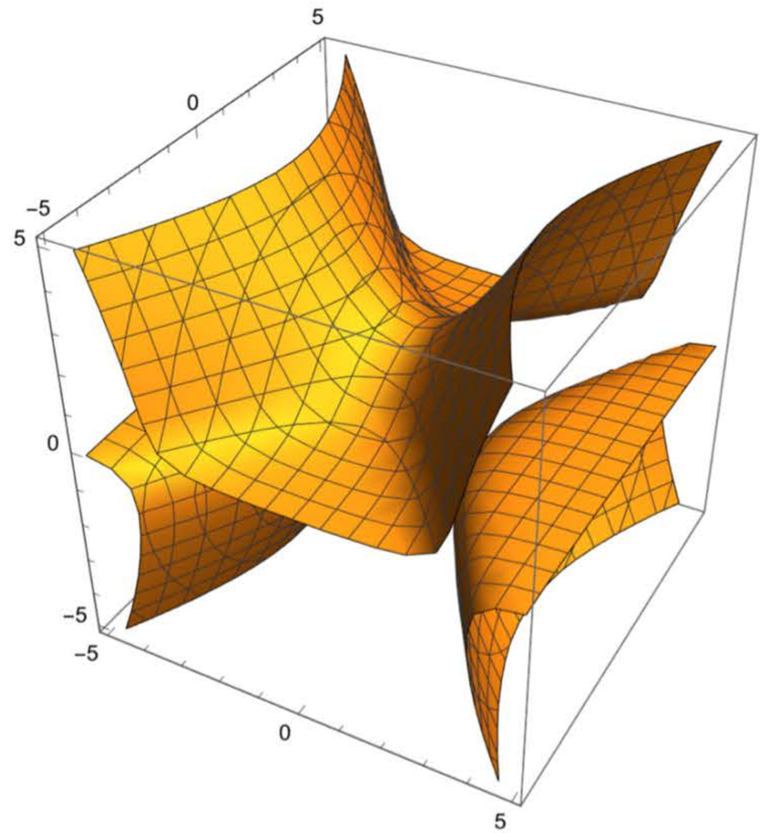


Figure 1: Variety with $C^3 = 1$. ($x_2 = x_3 = x_4 = x_5 = x_7 = 0$)

4. Reducible rep.

Thm. [Kirillov-Kostant-Souriau] (cpt case)

G: compact connected Lie group

λ: dominant integral highest weight

- The Coadjoint orbit $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ exists.
- The irred. rep $V_\lambda \leftarrow$ geom. quantization.

c.f.) $\mathcal{O}_\lambda$ is determined by all casimir op.

Traditionally understanding

$$\mathcal{O}_\lambda \xrightarrow[\text{quantize}]{} V_\lambda .$$

How can we interprete our results from this pt of view?

ex) $su(3)$

$[a,b]$: Dynkin idx

$\lambda = a\omega_1 + b\omega_2$ (fund. weight)

$= (\lambda_1, \lambda_2, \lambda_3)$

V_λ : (a,b) rep.

- $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$

$\mathcal{O}_\lambda = SU(3)/T^2$

$\lambda_1 = \lambda_2 \neq \lambda_3$

$\mathcal{O}_\lambda = \mathbb{C}P^2$

▷ Red rep.

$$V^{\mu} = \bigoplus_{a=1}^{m_\mu} V_\mu^a \quad , \quad P_a : V^\mu \to V_\mu^a \quad \text{projection}$$

$\rho_\mu^a : \mathfrak{g} \to \mathrm{End}(V_\mu^a)$: irreducible rep.

$\rho_\mu : \sum_{a=1}^{m_\mu} \rho_\mu^a P_a$, bose $\quad \hbar_a \rho_\mu^a(e_i) =: e_i^{(\mu,a)}$

$\hat{\hbar}_\mu := \hbar_1 \oplus \hbar_2 \oplus \cdots \oplus \hbar_m \qquad r_i \in \mathbb{R}$

$$[\rho_\mu(e_i), \rho_\mu(e_j)] = f_{ij}^k \rho_\mu(e_k)$$

$$[e_i^{(\mu)}, e_j^{(\mu)}] = \hat{\hbar}_\mu e_j^{(\mu)}$$

@ Casimir Op. (k-th)

not depend on "a"

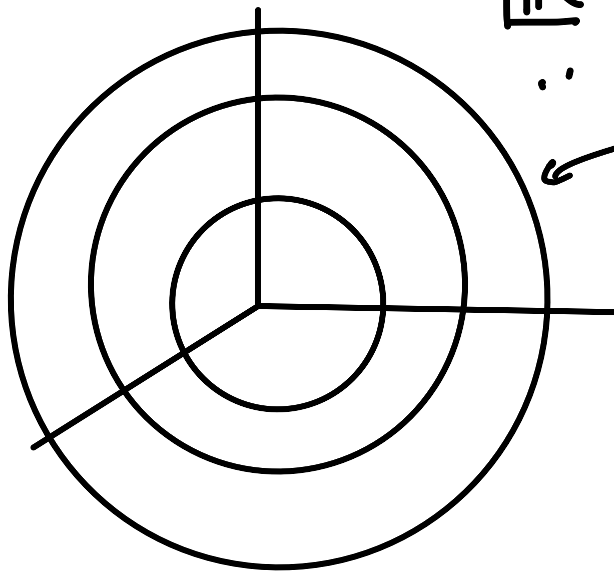
By $\quad C_i^{k,a}(e^{(\mu,a)}) = \hbar_a^k \Lambda_i^{a,k}(V_\mu^a)\, \mathrm{Id}_a = \lambda_i^k$

we fix each $\hbar_a$.

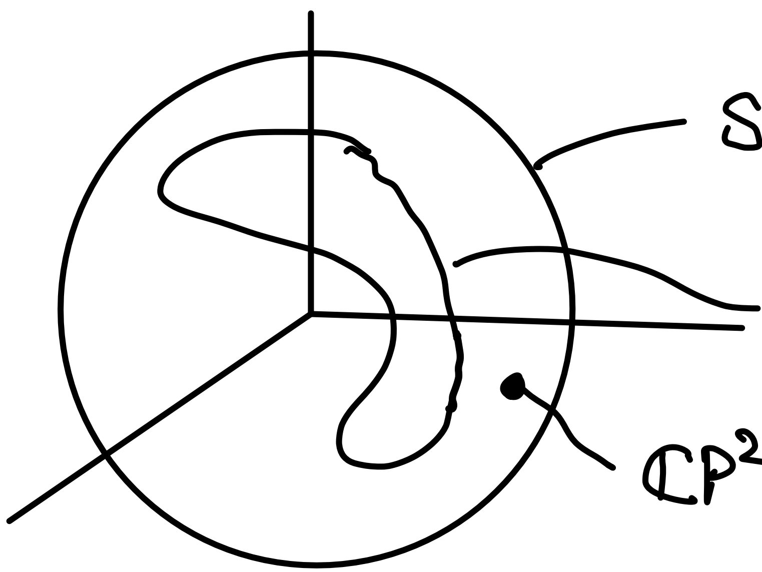
$\rightsquigarrow$ Everything extends to the case of red rep.

Singular foliation?

$\mathfrak{su}(2)$ case:



$\mathfrak{su}(3)$ case: $x_3 \sim x_8$ $\mathbb{R}^8$



5. Summary

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "g_\mu"] & gl(V^\mu) & & \\ \mathbb{C}[x]/I(C) \arrow[u, "I(C)=\{0\}"] \arrow[r, "g_{V/I}"] \arrow[rrr, bend left, "g_{A/I,\mu}"] & U_g[h]/I(C) \arrow[r, "\rho"] & gl(V^\mu) \arrow[r, "R"] & gl(V^\mu) \end{tikzcd}$$

rep

$\uparrow$ Casimir $C_i(x)$ $\qquad\qquad$ $\uparrow$ Casimir $g_\mu(C_i(x))$

$$f(x) = h(x) + r(x) \in \mathbb{C}[x]$$

$$\begin{aligned}[f(x)] &= [r(x)] \mapsto g_{V/I}([r(x)]) \mapsto \sum_I a_I e_I^{(\mu)} \mapsto \sum_{|I|<n_\mu} a_I e_I^{(\mu)} \\ &= \sum_I a_I [x_I] \qquad = \sum a_I [X_I]\end{aligned}$$

Not only reducible rep.

Future Works

- $\lim_{k\to\infty} 2\pi\hbar(N_k)\,\mathrm{Tr}(T_k(f)) = \int_M f\,\omega$ is unknown.
- Singularities in $\mathbb{C}[x]/I(C)$
- Singular Foliation ?
- Classical limit. Inverse problem.
- IKKT matrix model

Thank you for your attention !