代数多様体の行列正則化
～ファジー7次元球面とその一般化～

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1. Intro.

Mini Matrix model? (川合)

問 Lie代数の Large N 極限の古典論, 古典幾何は?

d-dim 半単純 Lie代数が解の行列模型

$$\boxed{S = -\frac{1}{4} g^{ij} g^{kl} \, \mathrm{tr}\, [X_i, X_k][X_j, X_l] + \frac{\hbar^2}{2} g^{ij} \, \mathrm{tr}\, X_i X_j}$$

$\rightsquigarrow$ E.O.M.

$$\underline{[X^i, [X_i, X_j]] = -\hbar^2 X_j}$$

X_i: Hermitian Mat.
$i = 1 \sim d$

Killing metric ~ diag(−1, 1, …, 1) の場合

(Ishibashi-Kawai-Kitazawa-Tsuchiya) (Yang-Sivakumar, Asano-Nishimura-Piensuk-Yamamori, etc.)

IKKT matrix model　mass-deformed

$$S = S_{IKKT} = \mathrm{tr}\left(-\frac{1}{4}[X_\mu, X_\nu]^2 + \hbar^2 \frac{1}{2} X^\mu X_\mu\right)$$

・行列模型の古典論を理解する上で重要.

▷ 非可換代数, 非可換幾何 $\rightleftharpoons$ 古典力学, 古典幾何
行列代数

パーシャル・シャンクスの定理

$\boxed{\text{多様体上の } C^\infty \text{ 関数環の同型} \simeq \text{多様体のトポロジーの同相}}$

$\rightsquigarrow$ 多様体を考える事と関数環考える事は同じ

古典的世界
可換なポアソン多様体
$(M, \cdot, \{\ ,\ \})$
M上の可換な関数環と$\{\ ,\ \}$

$\xrightarrow{\text{量子化}}$
$\xleftarrow[\text{古典極限}]{}$

高エネルギー
非可換代数,
(行列代数)
$[\ ,\ ]$

Lie-Poisson代数
$x^i \in \mathfrak{g}^* \quad \{x_i, x_j\} = f_{ij}^k x_k$

$\xrightarrow{\text{量子化}}$
$\xleftarrow[\text{古典極限}]{}$

Lie 代数
$X_i \in \mathfrak{g}, \ [X_i, X_j] = i f_{ij}^k X_k$

2. (弱)行列正則化の復習

c.f. Hasebe's talk.

▷ Fuzzy S^2, $\mathbb{R}^3$: J. Gohara, AS の方法に則して構成 (Madre, Hoppe)

su(2) case

$su(2)^* \simeq \mathbb{R}^3$

$(x_1, x_2, x_3) \in \mathbb{R}^3$ with su(2) Poisson str.

$$\{x_a, x_b\} = \epsilon^{abc} x_c$$

← Kirillov-Kostant $\mathfrak{g}(\mathfrak{g}^*) \to$ Poisson alg.

$A_{su(2)} = (\mathbb{C}[x], \cdot, \{,\})$: Poisson-Lie alg.

$f \in A_{su(2)}$: Polynomial on $\mathbb{R}^3$

$$f = f_0 + f_a x_a + \frac{1}{2} f_{ab} x_a x_b + \cdots$$

$f_{a_1 \cdots a_n}$: completely sym.

V^k : k-dim Vect. sp.

$\rho_k : A_{su(2)} \longrightarrow gl(V^k)$: linear

Step 1. $su(2)^* = \mathbb{R}^3$ の量子化

V^k : k-dim Vect. sp. (Lie代数の表現空間)

$g_k : A_{su(2)} \longrightarrow \mathrm{End}(V^k)$: linear

$$x^A = x_{a_1} \cdots x_{a_m} \mapsto g_k(x^A) = \begin{cases} \dfrac{\hbar^m}{m!} \displaystyle\sum_{\sigma \in \mathrm{Sym}(m)} J_{a_{\sigma(1)}} \cdots J_{a_{\sigma(m)}} & (m < k) \\ 0 & (m \geq k) \end{cases}$$

但し J_i $(i = 1 \sim 3)$ は $su(2)$ basis

$[J_i, J_j] = i\epsilon^{ijk} J_k$ (spin s, $\dim = 2s+1 = k$ rep.)

$$\boxed{g_k(x_i) = \hbar J_i}$$

$$[g_k(x_i), g_k(x_j)] = i\hbar f_{ij}^{\ k}\, g_k(x_k) + O(\hbar^2)$$

まだ S^2 の量子化ではない

$\hbar \to 0$ で可換

$\mathbb{R}^3$ 上の多項式環を量子化

Def). Casimir Polynomial

$\mathfrak{g}$: d-dim Lie alg.

$A_{\mathfrak{g}} := (\mathbb{C}[x], \cdot, \{\ ,\ \})$

$x = (x_1, \cdots, x_d)$

Kirillov-Kostant

$[X_i, X_j] = f_{ij}^k X_k$

$\{x_i, x_j\} = f_{ij}^k x_k$

Lie-Poisson alg.

$f(x)$ が Casimir poly. $\overset{\text{def}}{\Longleftrightarrow}$ $\{x_i, f(x)\} = 0$

$i = 1, 2, \cdots, d$

$su(2)$ には2次のカシミア多項式 $f^c = \delta^{ab} x_a x_b$

$$\mathcal{S}_k(f^c(x)) = \hbar^2 \delta^{ab} J_a J_b \quad : \text{Casimir op.}$$

$$= \hbar^2 \frac{1}{4}(k^2 - 1) \mathbb{1}_k$$

$$\hbar(g_k) = \sqrt{\frac{4\lambda^2}{k^2-1}} \Longleftrightarrow \delta^{ab} g_k(x_a)\, g_k(x_b) = \underline{\lambda^2}$$

fixed

$$\underbrace{\mathbb{R}^3}_{\text{ここを } S^2 \text{ に!!}} \xrightarrow{g_k} \mathrm{End}(V_k) \xrightarrow[\hbar \to 0]{k \to \infty} S^2$$

◎ function on S^2

$\mathbb{C}[x]$: $\mathbb{R}^3$ 上の多項式環

イデアル

$$I(C) := \{\, g(x)(\underbrace{\delta^{ab} x_a x_b - \lambda^2}_{\text{Eq of } S^2}) \mid g(x) \in \mathbb{C}[x] \,\}$$

i.e.) $I(C) \subset \mathbb{C}[x]$, module

$\forall f \in I(C)$, $\forall g(x) \in \mathbb{C}[x] \Rightarrow f(x)\, g(x) \in I(C)$

$\rightsquigarrow$ $\mathbb{C}[x] / I(C)$ が S^2 上の関数環(ポアソン代数)

Step 2. S^2 の量子化

$$g_{A/I,k} : \mathbb{C}[x] / I(C) \,(= S^2) \longrightarrow \mathrm{End}(V_k)$$

$$[f] = [r_f] \longmapsto \underline{g_{A/I,k}([f]) = g_k(r_f)}$$

Step 1 の量子化

$$f(x) = \underbrace{g(x)(\delta^{ab} x_a x_b - r^2)}_{\in\, I(C)} + \underbrace{r_f}_{I(C)\text{でわった余り}}$$

- well defined　Fuzzy S^2
- $[g_{A/I,k}([f]), g_{A/I,k}([g])] = i\hbar\, g_{A/I,k}(\{[f],[g]\}) + O(\hbar^2)$
- $g_{A/I,k}([f]\cdot[g]) = g_{A/I,k}([f]) \cdot g_{A/I,k}([g]) + O(\hbar)$

$\Downarrow$ 一般化できる

$\mathfrak{g}$ ($\dim \mathfrak{g} = d$)

▷ 半単純 Lie alg の カシミア多項式の空間の行列正則化
(Lie 群の作用で不変な空間)

$$\begin{tikzcd}
\mathbb{R}^d \simeq \mathfrak{g}^* \arrow[d] & \\
\mathbb{C}[x] \arrow[r, "g_\mu"] & \mathrm{End}(V^\mu) \\
\mathbb{C}[x]/I(C) \arrow[u, "I(C)=\{0\}"] \arrow[r, "g_{U/I}"] \arrow[rrr, bend left, "g_{\mathfrak{g}/I,\mu}"] & U_{\mathfrak{g}}[\hbar]/I(C) \arrow[r, "\rho", "\text{rep}"'] & \mathrm{End}(V^\mu) \arrow[r, "R"] & \mathrm{End}(V^\mu)
\end{tikzcd}$$

Casimir $C_i(x)$ Casimir $g_\mu(C_i(x))$

$f(x) = h(x) + r(x) \in \mathbb{C}[x]$

$[f(x)] = [r(x)] \mapsto g_{U/I}([r(x)]) \longmapsto \sum_I a_I e_I^{(\mu)} \longmapsto \sum a_I e_I^{(\mu)}$

$r(x) = \sum_I a_I x_I \qquad = \sum a_I [X_I] \qquad |I| < n_\mu$

$$\begin{tikzcd} \mathbb{C}[x]/I(C) \arrow[r, "g_{V/I}"] \arrow[rrr, bend left, "g_{A/I,\mu}"] & U_{\mathfrak{g}}[\hbar]/I(C) \arrow[r, "\rho_{V/I,\mu}"] & \mathrm{End}(V^{\mu}) \arrow[r, "R_{\mu}"] & \mathrm{End}(V^{\mu}) \end{tikzcd}$$

Casimir $C_i(x)$ Casimir $g_{\mu}(C_i(x))$

Def). $g_{A/I,\mu} : A_{\mathfrak{g}}/I(C) \longrightarrow \mathrm{End}(V^{\mu})$

$$g_{A/I,\mu} := R_{\mu} \circ g_{\mu}^{Pre} = R_{\mu} \circ \rho_{V/I,\mu} \circ g_{V/I}$$

Thm. $g_{A/I,\mu}$ is a weak M.R.

$$[g_{A/I,\mu}([f]), g_{A/I,\mu}([g])] = \hbar\, g_{A/I,\mu}(\{[f],[g]\}) + \hbar^2 P$$

Thm.

$$g_{A/I,\mu}([f]\cdot[g]) = g_{A/I,\mu}([f])\, g_{A/I,\mu}([g]) + \hbar P$$

問題 $\mathfrak{g}_{A/\mathbb{1},\mu}$ は、このままでは 忠実 ではない

ex). $\mathfrak{g} = \mathfrak{su}(3)$.

(p,q) rep.

fundmental weight vect.

∘ Highest weight $\mu = p\omega_1 + q\omega_2$, (p,q) : Dynkin label

2次 $C_2(x)$, 3次のカシミア多項式

$\mathfrak{g}_{A/\mathbb{1},(p,q)}(C_2(x)) = r^2 \,\mathrm{Id}$ となる様 $\hbar_{(p,q)} := \dfrac{r}{\sqrt{C_2(p,q)}}$

$\mathfrak{g}_{A/\mathbb{1},(p,q)}(C_3(x)) = \hbar^3 C_{3(p,q)}\,\mathrm{Id}$

$\rightsquigarrow \mathfrak{g}_{A/\mathbb{1},(p,q)}\big(\underbrace{C_3(x) - \hbar^3 C_3(p,q)}\big) = 0$

量子化 $\mathfrak{g}_{A/\mathbb{1},(p,q)}$ の Ker.

$C_2(x) = r^2$ (S^7) の量子化

を考えているのに.

$C_3(x) = \hbar C_3(p,q)$ も

$\rightsquigarrow$ $\boxed{\text{余随伴軌道}}$

[Kirillov-Kostant-Souriau]

余随伴軌道 $\xrightarrow[\text{量子化}]{\text{幾何学的}}$ Lie 代数の既約表現

実際

Prop. (AS)

$\mathfrak{g}$: 半単純 Lie alg. $\dim \mathfrak{g} = d$, $\operatorname{rank} \mathfrak{g} = r$

$$\begin{aligned} \rho_\mu : \mathbb{R}^d &\longrightarrow \operatorname{End} V^\mu \quad \text{既約表現} \\ x_i &\longmapsto e_i^\mu \end{aligned}$$

$\hbar(\mu)$ をある k で $\hbar^k C_k(x) = \lambda$ で固定し.

$\dim V^\mu \longrightarrow \infty$ $(\hbar \to 0)$ となる表現の列で

ρ_μ の列を決めると. 一般の配位で

$$\lim_{\hbar \to 0} \hbar^i C_i(x) = \lambda_i \quad (i = 1, \cdots, r)$$ が決まる.

$\rightsquigarrow$ 既約表現の行列正則化は. 余随伴軌道に

3. 可約表現で埋める.

$$V^{\mu} = \bigoplus_{a=1}^{m_\mu} V_\mu^a \quad , \quad P_a : V^\mu \to V_\mu^a \quad \text{projection}$$

$\rho_\mu^a : \mathfrak{g} \longrightarrow \mathrm{End}(V_\mu^a)$: irreducible rep.

$\rho_\mu : \sum_{a=1}^{m_\mu} \rho_\mu^a P_a$, base $\quad \hbar_a \, \rho_\mu^a(e_i) =: e_i^{(\mu,a)}$

$$\hat{\hbar}_\mu := \hbar_1 \oplus \hbar_2 \oplus \cdots \oplus \hbar_m \qquad r_i \in \mathbb{R}$$

$$[\rho_\mu(e_i), \rho_\mu(e_j)] = f_{ij}^k \rho_\mu(e_k)$$

$$[e_i^{(\mu)}, e_j^{(\mu)}] = \hat{\hbar}_\mu e_j^{(\mu)}$$

not depend on "a"

@ Casimir Op. (k-th)

By $\quad C_i^{k,a}(e^{(\mu,a)}) = \hbar_a^k \Lambda_i^{a,k}(V_\mu^a)\, \mathrm{Id}_a = \lambda_i^k$

we fix each $\hbar_a$.

$\rightsquigarrow$ Everything extends to the case of red rep.

Def) $g^R_{A/I,\mu} : A_{\mathfrak{g}}/I(C) \to gl(V^\mu) = gl\left(\bigoplus_{a=1}^{m_\mu} V^a_\mu\right)$ 14.

$$g^R_{A/I,\mu} = R^R_\mu \circ \rho^R_{U/I,\mu} \circ g^R_{U/I}$$

$$\mathbb{C}[x]/I(C) \xrightarrow{g^R_{U/I}} \begin{matrix} U_{\mathfrak{g}}[\hbar_1]/I(C) \\ \oplus \\ \vdots \\ \oplus \\ U_{\mathfrak{g}}[\hbar_{m_\mu}]/I(C) \end{matrix} \xrightarrow{\rho^R_{U/I,\mu}} \begin{matrix} gl(V^\mu_1) \\ \oplus \\ \vdots \\ \oplus \\ gl(V^\mu_{m_\mu}) \end{matrix} \xrightarrow{R^R_\mu} gl(V^\mu)$$

$$[f] = [r_f] \mapsto \begin{matrix} [g_U(r_f)|_{\hbar=\hbar_1}] \\ \oplus \\ \vdots \\ \oplus \\ [g_U(r_f)|_{\hbar=\hbar_{m_\mu}}] \end{matrix} \mapsto \begin{matrix} \sum_J a^{(J)} e^{(\mu,1)}_{(J)} \\ \oplus \\ \vdots \\ \oplus \\ \sum_J a^{(J)} e^{(\mu,m_\mu)}_{(J)} \end{matrix} \mapsto \begin{matrix} \sum_{|J|\le \eta^1_\mu} a^{(J)} e^{(\mu,1)}_{(J)} \\ \oplus \\ \vdots \\ \oplus \\ \sum_{|J|\le \eta^{m_\mu}_\mu} a^{(J)} e^{(\mu,m_\mu)}_{(J)} \end{matrix}$$

Then.

$$\boxed{\begin{aligned} &[g^R_{A/I,\mu}([f]),\, g^R_{A/I,\mu}([g])] = \hat{\hbar}(\mu)\, g^R_{A/I,\mu}(\{[f],[g]\}) + O(\hbar^2) \\ &g^R_{A/I,\mu}([f][g]) = g^R_{A/I,\mu}([f]) \cdot g^R_{A/I,\mu}([g]) + O(\hbar) \end{aligned}}$$

Ex.) Fuzzy $\mathbb{R}^3$

$$\mathcal{g}_{k,r} : A_{su(2)} \longrightarrow \mathrm{Mat}_k(\mathbb{C})$$

$$x_{a_1}\cdots x_{a_m} \longmapsto \frac{\hbar^m(k,r)}{m!} \sum_{\sigma\in S_m} J_{a_{\sigma(1)}}\cdots J_{a_{\sigma(m)}}$$

where J_a : a generator for k-dim irre. rep of $su(2)$

$$\delta^{ab} J_a J_b = \frac{1}{4}(k^2-1)\, \mathbb{1}_k ,$$

$\hbar(k,r)$ is determined by

$$\underline{\hbar^2(k,r)\ \frac{1}{4}(k^2-1) = r^2}$$

Consider $\mathcal{g}^R_k := \bigoplus_{i=1}^{k^2} \mathcal{g}_{k,r_i}$. $r_1 = \sqrt{\frac{1}{k}},\ r_2 = \sqrt{\frac{2}{k}}, \cdots, r_{k^2} = \sqrt{\frac{k^2}{k}} = \sqrt{k}$

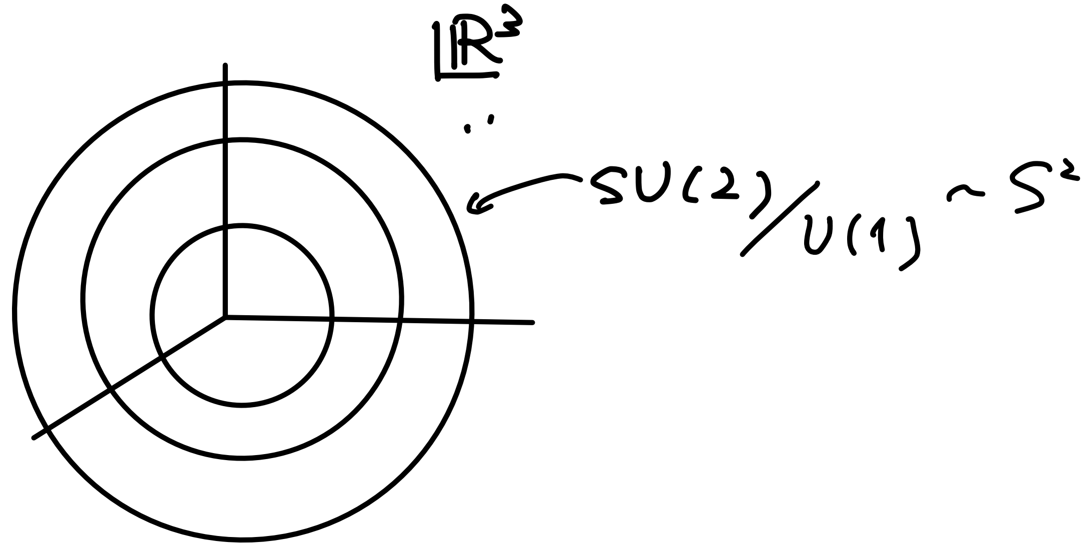
When $k\to\infty$, $r_{k^2}\to\infty$, $r_1 \to 0$, $\underbrace{r_{i+1}-r_i \to 0}_{\text{稠密}}$

(commutative lim $\hbar\to 0$)

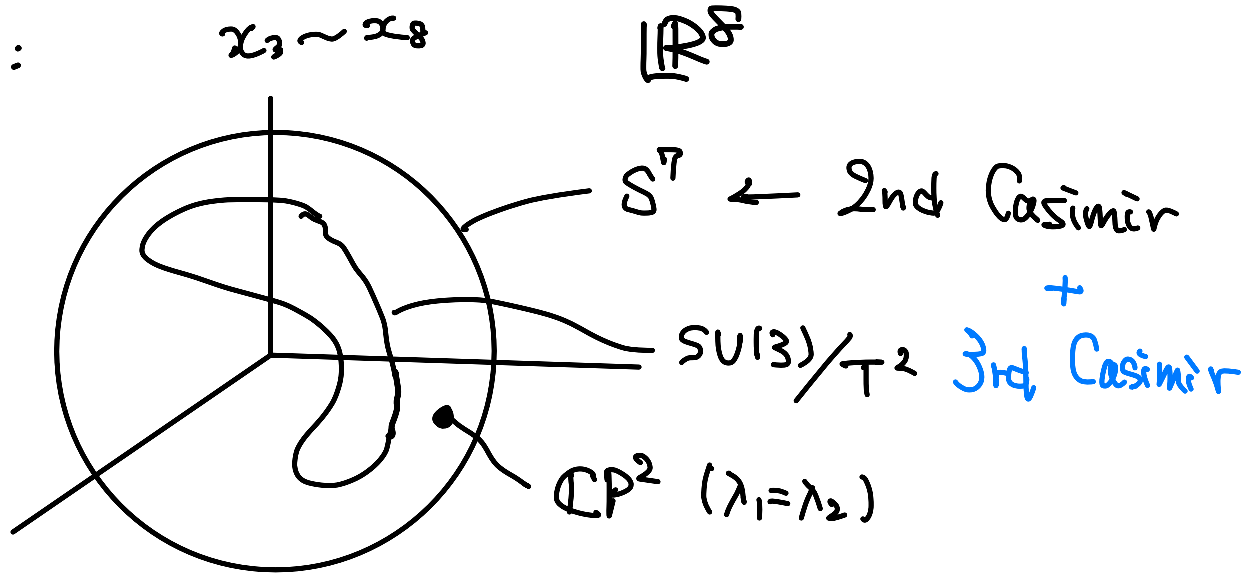
<u>$\mathbb{R}^3$ is densely foliated by S^2</u>

Singular foliation

$\mathfrak{su}(2)$ case :



$\mathfrak{su}(3)$ case :



4. Fuzzy S^7

$\mathfrak{g} = su(3)$.

$\triangleright$ (p,q) rep.

fundmental weight vect.

- Highest weight $\mu = p\,\omega_1 + q\,\omega_2$, (p,q) : Dynkin label

$$\dim V_{(p,q)} = \frac{1}{2}(p+1)(q+1)(p+q+2)$$

$$C_2(p,q) = \frac{1}{3}(p^2+q^2+pq) + (p+q) \geq pq$$

$$C_3(p,q) = \frac{1}{18}(p-q)(2p+q+3)(p+2q+3)$$

Step 1. Fuzzy $\mathbb{R}^8$

Let's def. $\mathcal{g}^r_{p,q} : \underbrace{A_{su(3)}}_{\simeq \mathbb{R}^8} \longrightarrow gl(V_{(p,q)})$, similarly.

$$\hbar_{(p,q)} := \frac{r}{\sqrt{C_2(p,q)}}$$

r : some fixed constant.

Lem. 既約表現の列 (p_n, q_n) s.t. $\lim_{n\to\infty} \frac{p_n}{q_n} = t \ (0 \le t)$

$$\lim_{n\to\infty} g_\mu(C_3(x)) = \left(\lim_{n\to 0} \hbar^3(p_n, q_n)\, C_3(p_n, q_n)\right) \mathrm{Id} = r^3 R(t)\, \mathrm{Id}$$

where $-\frac{\sqrt{3}}{3} \le R(t) \le \frac{\sqrt{3}}{3}$

- $R(0) = -\frac{\sqrt{3}}{3}$
- $R(\infty) = \frac{\sqrt{3}}{3}$
- $R(t)$: 単調増加

Lem.

$$\mathcal{O}_R = \left\{ x \in \mathbb{R}^8 \;\middle|\; \sum_{i=1}^{8} x_i^2 = r^2, \quad C_3(x) = r^3 R \right\}$$

Let D be a dense subset of $\left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$. $\mathcal{D} := \coprod_{R\in D} \mathcal{O}_R$

Then, $\overline{\mathcal{D}} = S^7$

$$C^3 = \frac{1}{18}(2\sqrt{3}x_8^3 - 6\sqrt{3}x_1^2x_8 - 6\sqrt{3}x_2^2x_8 - 6\sqrt{3}x_3^2x_8 + 3\sqrt{3}x_4^2x_8 + 3\sqrt{3}x_5^2x_8 + 3\sqrt{3}x_6^2x_8 + 3\sqrt{3}x_7^2x_8$$
$$- 18x_2x_5x_6 + 18x_2x_4x_7 - 18x_1(x_4x_6 + x_5x_7) - 9x_3\left(x_4^2 + x_5^2 - x_6^2 - x_7^2\right)).$$

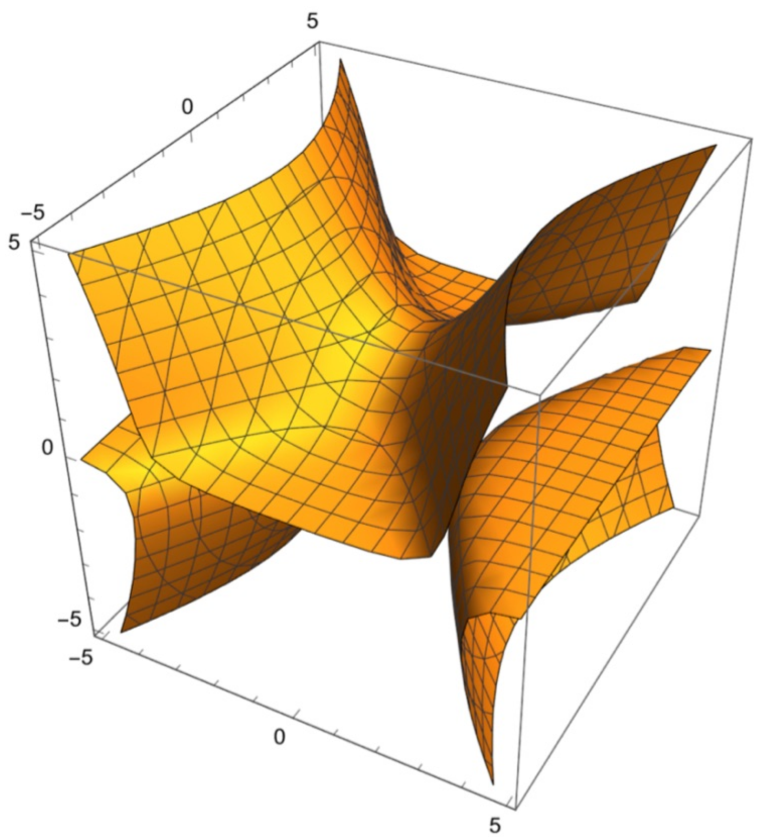


Figure 1: Variety with $C^3 = 1$. $(x_2 = x_3 = x_4 = x_5 = x_7 = 0)$

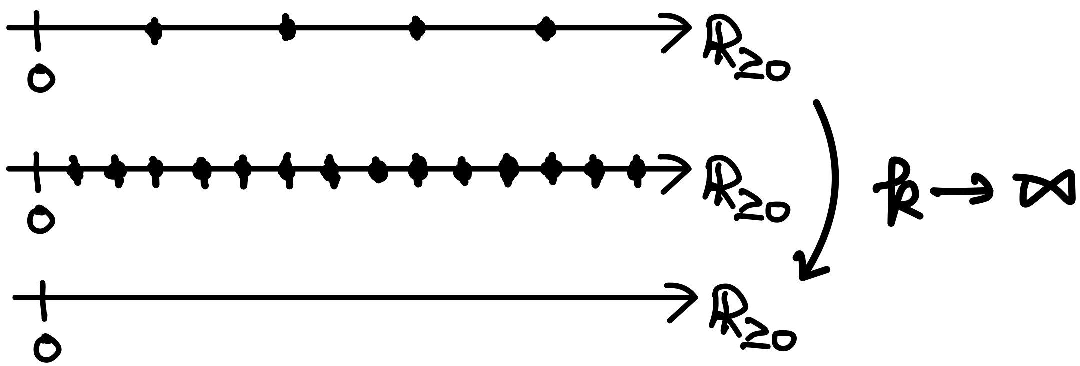
Step2. $I(C)(x) = \left((C_2(x) - r) \right)$

$g_{A/I,(p_n,q_n)}([f]) = g^{r}_{p_n,q_n}(r_f)$ (If $r_f \leq n_{(p,q)}$)

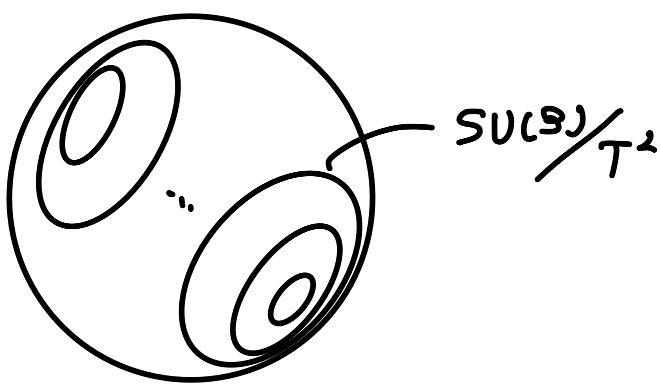
$f = h + r_f \quad h \in I(C)(x)$

Step3. $\{ (p_n, q_n) \mid p_n + q_n = k \}$

$t_n = \frac{p_n}{q_n} = 0, \frac{1}{k-1}, \cdots, k-1, \infty \rightsquigarrow$ t_n は半直線 $\cup \{\infty\}$ で $k \to \infty$ で稠密になる



Step4. $g^{R}_{A/I,k} := \bigoplus_{p_n + q_n = k} g_{A/I,(p_n,q_n)} \qquad k = 1, 2, \cdots$



$\xrightarrow{\hbar \to 0} S^7$

5. 一般化

Thm $\mathfrak{g}$: cpt 半単純, $\dim\mathfrak{g} = d$, $\mathrm{rank}\,\mathfrak{g} = r$.

$C^n(x)$: d次のカシミア $n = 1, 2, \cdots, r$

$S_k := \{ x \in \mathbb{R}^d \mid C^k(x) - \lambda_k = 0 \}$.

$\mathbb{C}[x]/I(C)$: S_k 上の関数環.

$g_{S_k,\mu} : \mathbb{C}[x]/I(C) \longrightarrow \mathrm{End}(V^\mu)$: S_kの量子化で

$\hbar \to 0$ ($\dim V^\mu \to \infty$) の古典極限が S_k

であるものが存在

㊟ 余随伴軌道が S_k を稠密に埋めつくすという意味

6. Summary

$$\begin{tikzcd}
\mathbb{C}[x]/I(C) \arrow[r, "\rho^R_{V/I}"] \arrow[rrr, bend left, "\rho^R_{A/I,\mu}"] & \begin{matrix} U_{\hbar}\mathfrak{g}[\hbar_1]/I(C) \\ \oplus \\ \vdots \\ \oplus \\ U_{\hbar}\mathfrak{g}[\hbar_{m_\mu}]/I(C) \end{matrix} \arrow[r, "\rho^R_{V/I,\mu}"] & \begin{matrix} gl(V_1^\mu) \\ \oplus \\ \vdots \\ \oplus \\ gl(V^\mu_{m_\mu}) \end{matrix} \arrow[r, "R^R_\mu"] & gl(V^\mu)
\end{tikzcd}$$

↑ an Algebraic variety defined by inv. polynomial of $\mathfrak{g}$. (ex. S^7) → (Fuzzy S^7)

↑ A Irreducible rep. each rep ~ Coadjoint orbit.

- In $\dim V^\mu \to \infty$
- $\hbar \to 0$ (commutative alg).
- The increasing family of subvarieties (Coadjoint orbits) becomes dense in the original variety.
- We can generalize S^7 case to $\forall\, C_i(x) = \lambda_i$ (Casimir) in a compact semisimple Lie alg. case.

ご清聴ありがとうございます

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