テンソルの固有値・固有ベクトル分布の普遍性

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§Eigenvalues/vectors of matrices and tensors

Matrix: M_{ab} $a, b = 1,2,\cdots,N$ N : dimension

Eigen equation

$$M_{ab}v_b = \zeta\, v_a$$

ζ : eigenvalue

v_a : eigenvector

Appears in various linear contexts

Tensor : $T_{a_1 a_2 \cdots a_p}$ $\quad p \geq 3 \quad a_i = 1,2,\cdots,N \quad N$: dimension

Tensor eigen equation (typical form)

$$T_{a_1 a_2 a_3 \cdots a_p} v_{a_2} v_{a_3} \cdots v_{a_p} = \zeta\, v_{a_1} \qquad (|v| = 1)$$

ζ : eigenvalue

v_a : eigenvector

A modern generalization of eigenvalues/eigenvectors to tensors

Qi,2005, Lim, 2005

Appears in various non-linear contexts

Properties largely different from the matrix case

Number of eigenvalues $\sim e^{cN}$: exponential in N

Cartwright, Sturmfels, 2013

Computing eigenvalues/vectors is extremely hard (NP-hard)

Hillar, Lim, 2009

§Random matrices

Quantum mechanics

$$\hat{H}|\psi^{(i)}\rangle = E^{(i)}|\psi^{(i)}\rangle$$

$$i = 0,1,\cdots$$

$E^{(i)}$: energy eigenvalues (spectra)

$|\psi^{(i)}\rangle$: energy eigenstates

Matrix eigen problem

$$M_{ab}v_b = \zeta\, v_a \qquad (N = \infty)$$

$$\zeta = E^{(i)}$$

$$v_a = \langle a|\psi^{(i)}\rangle$$

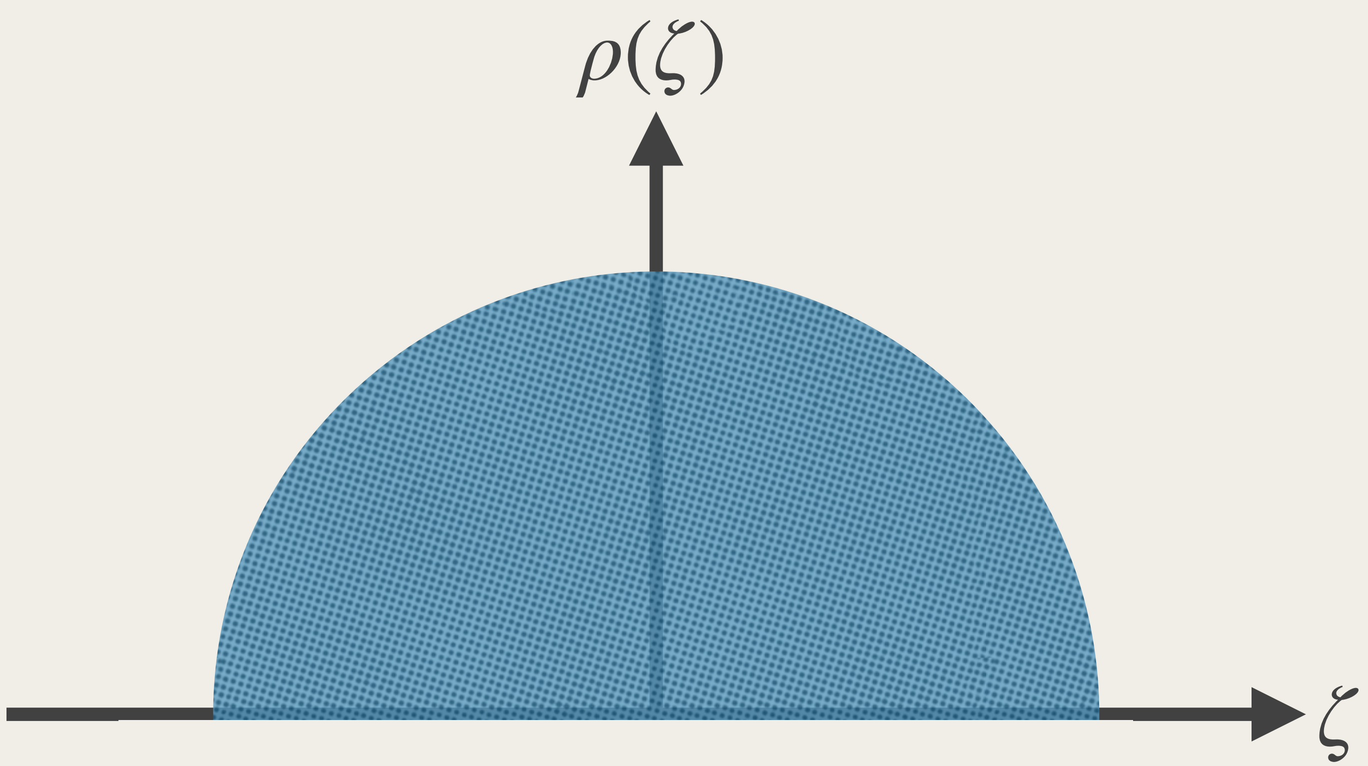
$$M_{ab} = \langle a|\hat{H}|b\rangle$$

$|a\rangle$: Hilbert space basis

Wigner 1955

Energy spectra of heavy nuclei may be computed by considering random matrix (random value for each entry $M_{ab} = M^*_{ba}$)

In $N \to \infty$, the eigenvalue distribution has a semi-circle shape



Wigner's semi-circle law

$$M_{ab} v_b = \zeta\, v_a$$

$M_{ab} = M^*_{ba} = \mathcal{N}(0,1)$

Normal distribution

$\rho(\zeta)$: eigenvalue density

Various universalities are known for eigenvalue distributions of random matrices

Bulk universalities :

- Wigner's semi-circle distribution appears for

  M : real symmetric, or hermitian

  Each entry M_{ab} can be different from normal distribution

- Universality of correlation functions of eigenvalue distributions
- Gap universalities between adjacent eigenvalues

Edge universality :

- Universality of the distribution of the largest (lowest) eigenvalue
  Tracy-Widom distribution

These universalities support wide applicability in various subjects

Quantum chromo dynamics, solid state physics, quantum chaos, quantum optics, statistics, neural networks, deep learning, numerical analysis, number theory (Riemann hypothesis), free probability, computational neuroscience, optimal control, computational mechanics, engineering, and more.

§Random tensors

It was originally started as a model of emergent spacetime in QG.
Ambjorn, Durhuus, Jonsson, 1991, NS,1991, Godfrey, Gross, 1991, Gurau, 20009

But we here focus on contexts of eigenvalues/vectors.

(Ex. 1) Spherical p-spin model for spin glasses   Crisanti, Sommers, 1991

$$H = T_{a_1 a_2 \cdots a_p} v_{a_1} v_{a_2} \cdots v_{a_p} \qquad |v| = 1$$

$T_{a_1 a_2 \cdots a_p}$ is randomly provided (ex. $\mathcal{N}(0,1)$)

A popular model for spin glasses (largely analytically computable)

Stationary points of H are determined by tensor eigen equation

$$T_{a_1 a_2 \cdots a_p} v_{a_2} \cdots v_{a_p} = \zeta \, v_{a_1} \qquad |v| = 1$$

The energy landscape is very rough $\sim e^{cN}$ — Complexity

(Ex. 2) Entanglement of quantum states

$$|\psi\rangle = T_{a_1a_2\cdots a_p}|a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_p\rangle$$

How much entangled is the quantum state $|\psi\rangle$?

The geometric measure — Distance from product states
Shimony, 1995
(Zero entangled states)

The distance is determined by the largest eigenvalue of the tensor $T_{a_1a_2\cdots a_p}$

Typical quantum states have the amount of entanglement determined by the edge of the eigenvalue distribution of random tensors

§Eigenvalue/vector distributions of random tensors

Eigenvalue/vector distributions of random tensors have applications in spin glasses, quantum information theory, artificial intelligence, general relativity, string theory, etc.

The range of applications will continue growing in various non-linear contexts, as random matrices in linear contexts.

Until very recently, knowledge had been quite limited, due to technical complications.

An important question would be whether there is "universality", since this assures wide applicabilities

We recently developed quantum field theoretical methods to systematically compute them, and have found a universality (though the most basic one)

§A universality

There exist various tensor eigen equations, depending on properties of tensors and eigenvalues/vectors, more than matrix cases

(1) $$T_{a_1a_2\cdots a_p}v_{a_2}v_{a_3}\cdots v_{a_p}=\zeta\, v_{a_1}$$

$T_{a_1a_2\cdots a_p}$: symmetric real tensor v_a : real vector ζ : real

$(T_{a_1a_2\cdots a_p}=T_{a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(p)}}$ for arbitrary permutations $\sigma)$

(2) $$T_{a_1a_2\cdots a_p}v^*_{a_2}v^*_{a_3}\cdots v^*_{a_p}=\zeta\, v_{a_1}$$ $*$: complex conjugation

$T_{a_1a_2\cdots a_p}$: symmetric complex tensor v_a : complex vector ζ : real

(3) $$T_{a_1a_2\cdots a_p}v^{(2)}_{a_2}v^{(3)}_{a_3}\cdots v^{(p)}_{a_p}=\zeta\, v^{(1)}_{a_1}$$

$$T_{a_1a_2\cdots a_p}v^{(1)}_{a_1}v^{(3)}_{a_3}\cdots v^{(p)}_{a_p}=\zeta\, v^{(2)}_{a_2}$$

$$\vdots$$

$$T_{a_1a_2\cdots a_p}v^{(1)}_{a_1}v^{(2)}_{a_2}\cdots v^{(p-1)}_{a_{p-1}}=\zeta\, v^{(p)}_{a_p}$$

$T_{a_1a_2\cdots a_p}$: real tensor (each index is independent)

There are p real eigenvectors : $v^{(i)}_a$ $(i=1,2,\cdots,p)$

$\rightarrow$ A system of p eigen equations

ζ : real

(4) $$T_{a_1 a_2 \cdots a_p} v_{a_2}^{(2)*} v_{a_3}^{(3)*} \cdots v_{a_p}^{(p)*} = \zeta\, v_{a_1}^{(1)}$$

$$T_{a_1 a_2 \cdots a_p} v_{a_1}^{(1)*} v_{a_3}^{(3)*} \cdots v_{a_p}^{(p)*} = \zeta\, v_{a_2}^{(2)}$$

$$\vdots$$

$$T_{a_1 a_2 \cdots a_p} v_{a_1}^{(1)*} v_{a_2}^{(2)*} \cdots v_{a_{p-1}}^{(p-1)*} = \zeta\, v_{a_p}^{(p)}$$

$T_{a_1 a_2 \cdots a_p}$: complex tensor (each index is independent)

p complex eigenvectors : $v_a^{(i)}$ $(i = 1,2,\cdots,p)$

ζ : real

$$(5) \quad \epsilon_{i_1 i_2 \cdots i_p} T_{a_1 a_2 \cdots a_p} v_{a_2}^{(i_2)} v_{a_3}^{(i_3)} \cdots v_{a_p}^{(i_p)} = \zeta \, v_{a_1}^{(i_1)}$$

$T_{a_1 a_2 \cdots a_p}$: anti-symmetric real tensor

$\epsilon_{i_1 i_2 \cdots i_p}$: Levi-Civita symbol, $\epsilon_{12 \cdots p} = 1$

There are p real eigenvectors $v_a^{(i)}$ $(i = 1,2,\cdots,p)$ which form an orthonormal system : $v^{(i)} \cdot v^{(j)} = \delta_{ij}$

ζ : real eigenvalue

A system of p eigen equations

$$(6) \quad \epsilon_{i_1 i_2 \cdots i_p} T_{a_1 a_2 \cdots a_p} v_{a_2}^{(i_2)*} v_{a_3}^{(i_3)*} \cdots v_{a_p}^{(i_p)*} = \zeta \, v_{a_1}^{(i_1)}$$

$T_{a_1 a_2 \cdots a_p}$: anti-symmetric complex tensor

p complex eigenvectors $v_a^{(i)}$ $(i = 1,2,\cdots,p)$ with $v^{(i)*} \cdot v^{(j)} = \delta_{ij}$

ζ : real

And there are more ….

The eigen equations (1)-(6) have no mutual relations.

However, for random tensors with normal distributions, we found a universal asymptotic formula of the eigenvalue distributions in $N \to \infty$

Delporte, La Scala, Sasakura, Toriumi, 2025

$$\rho(\zeta) \sim \exp\left[N\, d\, h_p\left(\zeta_c^2/\zeta^2\right)\right]$$

$$h_p(x) := \mathrm{Re}\left[\frac{1}{2}\log(p-1) + \frac{1}{x}\left(-1 + \frac{2}{p} - \sqrt{1-x}\right) + \frac{1}{2}\log x - \log\left(1 - \sqrt{1-x}\right)\right]$$

d : total degrees of eigenvectors

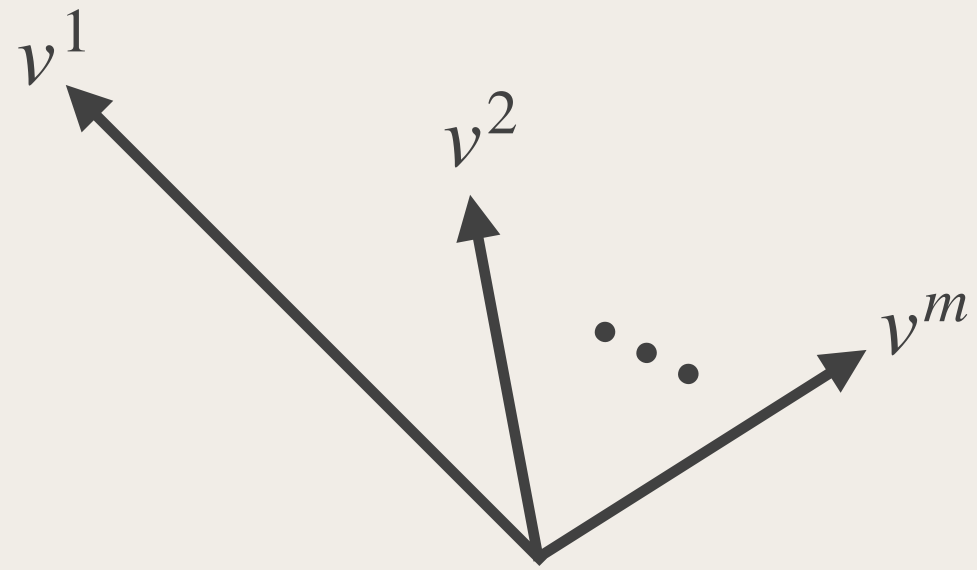
(1) 1 (2) 2 (3) p (4) $2p$ (5) p (6) $2p$

ζ_c : determined by the phase transition point of the quantum field theory associated to each eigen problem

We also found a universal asymptotic formula of joint distributions of eigenvectors for $N \to \infty$

NS, 2026

$$\rho(v^1, v^2, \cdots, v^m)$$



Currently for (1), (2) only, the other cases being under study

§Brief sketch of the method of computations

We use quantum field theoretical methods

For simplicity, symmetric real tensor case

$$\rho(v) = \left\langle \left|\det J\right| \prod_{a=1}^{N} \delta(f_a) \right\rangle_T$$

$$f_a = v_a - T_{a_1 a_2 a_3 \cdots a_p} v_{a_2} v_{a_3} \cdots v_{a_p}$$

$$J_{ab} = \frac{\partial f_b}{\partial v_a}$$

$\zeta = 1$ is chosen instead of $|v| = 1$

$\langle \cdot \rangle_T$: average over random $T_{a_1 a_2 \cdots a_p}$ (normal distribution)

Noting $|\det J| = \lim_{\epsilon \to +0} \dfrac{\det(J^2 + \epsilon I)}{\sqrt{\det(J^2 + \epsilon I)}}$

$\rho(v)$ can be rewritten in a quantum field theoretical way

$$\rho(v) = \lim_{\epsilon \to +0} \int dT d\lambda d\Phi \, e^S$$

$$S = -\alpha |T|^2 + i\lambda \cdot f + i\, G_{\kappa\kappa'} \Phi_\kappa \cdot J \cdot \Phi_{\kappa'} + \sqrt{\epsilon}\, \tilde{G}_{\kappa\kappa'} \Phi_\kappa \cdot \Phi_{\kappa'}$$

$G_{\kappa\kappa'}$, $\tilde{G}_{\kappa\kappa'}$: certain constant matrices

Φ_κ : superfield (combo of bosonic and fermionic variables)

$T_{a_1 a_2 \cdots a_p}$ and λ_a can be integrated explicitly, because they appear at most quadratically.

After T, λ integration

$$\rho(v) = \cdots \, Z_\Phi$$

$$Z_\Phi = \int d\Phi \, e^{S_\Phi}$$: quantum field theory of superfield

$$S_\Phi \sim -\Phi\Phi - g(v)\,\Phi\Phi\Phi\Phi$$: quartic interactions

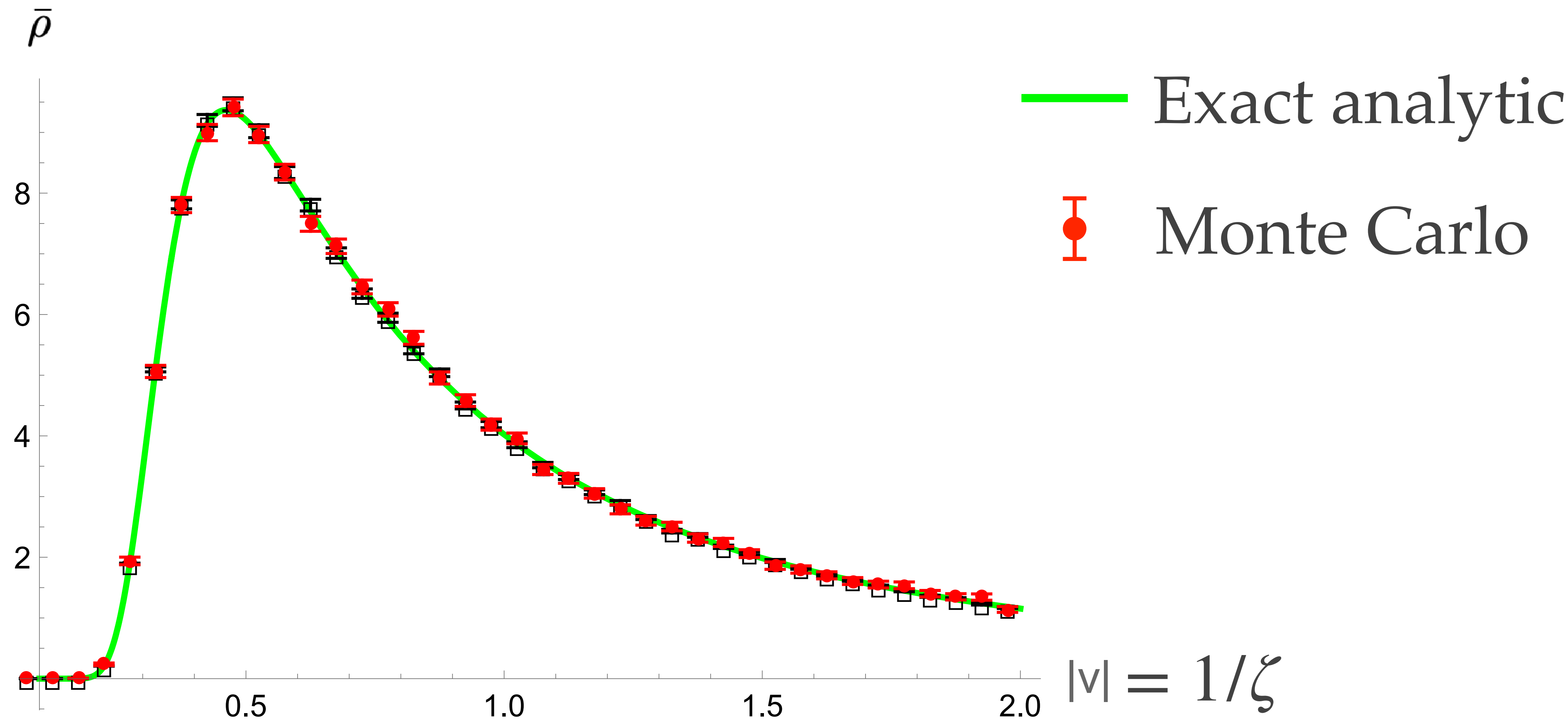
The quantum field theory cannot be computed exactly in general.

However, in $N \to \infty$, it can be solved by a method using Schwinger-Dyson equations.

This quantum field theory has a phase transition point separating weak and strong coupling regimes in $N \to \infty$.

This phase transition point determines the rescaling parameter ζ_c in the universal formula

Comparison with Monte-Carlo



$p = 3,\ N = 4, \alpha = 1/2$

§ Summary and future problems

Universalities assure wide applicability.

We found a universality of tensor eigenvalue / vector distributions, which holds across various eigen problems.

However, the result is still quite limited, compared with the universalities known in random matrices, and should be extended.

As future study, it would be wanted to

- Complete universality classifications
- Generalize randomness from normal distributions
- Study edge universalities

and more