Emergent Quantum Matrix Geometry: Matrix first or State first ?

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[Refs.] arXiv 2604.24112 (to appear in PRD) and references therein.

Geometry from Quantum State

Standard scheme

$$\{x, y\} \Rightarrow [\widehat{x}, \widehat{y}]$$

Symplectic manifold → Quantization of symplectic manifold

This is the basic idea behind the conventional quantization methods:

Weyl quant., deformation quant., geometric quant., BT quant. ..

(except for Connes' NCG)

Restricted to symplectic manifolds → General manifold ?

Restricted to the commutator formalism → General NCG such as Nambu bracket ?

Classical Geometry ⇒ Quantum Geometry

Present scheme

Quantum State ⇒ Matrix Geometry

(Connes' NCG: Algebra → Geometry)

Quantum mechanics and M(atrix) theory

This year is the centennial of wave mechanics!

Matrix mechanics (1925) ⟷ Wave mechanics (1926)

Matrix ***Quantum State, Operator***

(from wiki)

$$F^{kl} = \int \rho(x) u_k(x) [F, u_l(x)] dx,$$

Schroedinger (Annalen der Physik *1926*)

(from wiki)

$$\boxed{O_{ij} = \langle \psi_i | O(x, p = -i\partial_x) | \psi_j \rangle}$$

Matrix (operator) first or state first?

M(atrix) theory in string theory

IKKT('97) *BFSS ('97)* $\cdots$

$$S = \frac{1}{4}\mathrm{tr}([A_\mu, A_\nu]^2) + \cdots$$

(IIB or not IIB ?)

Matrix geometry from Quantum states

Matrix model

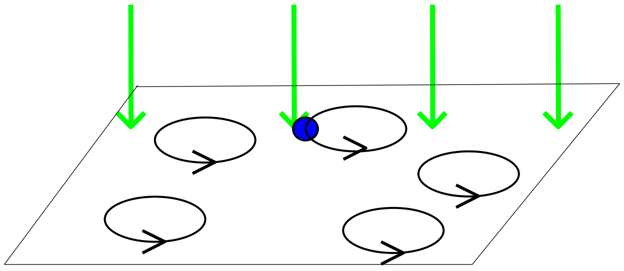
(Matrix) Geometry (classical sol.) ⟹ Fluctuation (field theory) ⟹ Quantum states

True quantum vac.
(*Nishimura et al*.)

Quantum matrix geometry

= Matrix geometry
from quantum states

Landau model



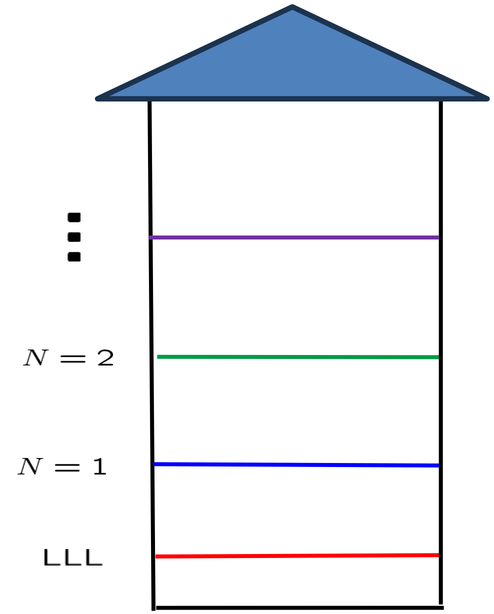
Level projection ⟹ Quantum states ⟹ (Matrix) Geometry

My past studies

‘02~ LLL geometry (higher D, SUSY …) ``First floor”

Level projection for matrix geo.

⇨ *``Toy model” of Matrix model geometry*



Explore Landau models to generate a new NCG !

’16: 2D higher LLs & fuzzy S2 ``Upper floors”

’17: fuzzy S3 & higher LLs

’20: fuzzy S4 & higher LLs

⇨ *Nested Nambu geometry*

Ishiki’s group (‘18~)

Tsuchiya’s group (‘20~)

’23: Prescription for general G/H

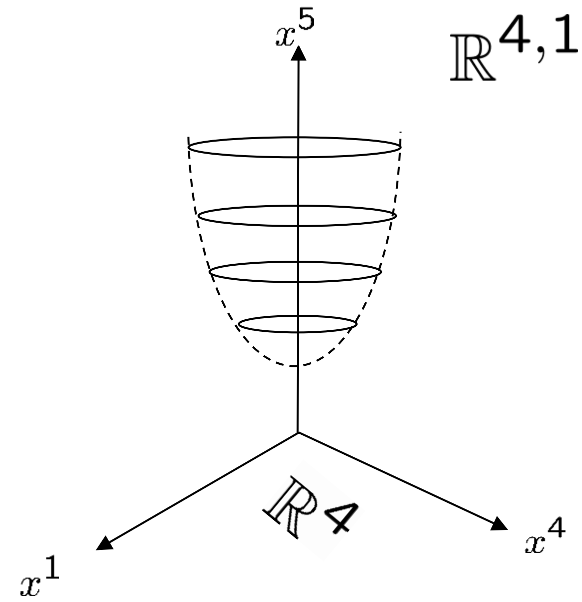
’25: Deformed fuzzy manifold

``Relation between all floors”

’26: Howe duality

Roger Howe (from wiki)

Emergent Quantum Matrix Geometry I



$\boxed{\text{Higher D fuzzy hyperboloid}}$

Lowest Landau level on 4D hyperboloid

➡ 4D fuzzy hyperboloid $(\mathbb{C}P^{1,2}_{\text{fuzzy}})$
(*KH '12*)

Higher D. indefinite signature fuzzy manifold

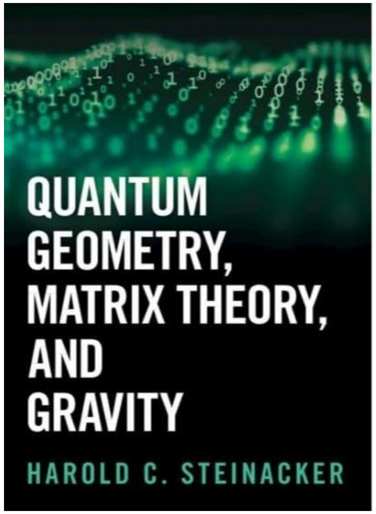
$$[X^a, X^b] = iX^{ab} \quad [X^a, X^{bc}] = i\eta^{ab}X^c - i\eta^{ac}X^b \quad \cdots$$

Classical sol. of indefinite signature Matrix model with mass term

$$\eta_{bc}[X^b, [X^c, X^a]] = M^2 X^a$$

➡ Emergent gravity from IKKT (*Steinacker '18*~)
(FRW cosmology)

Chap.5 in



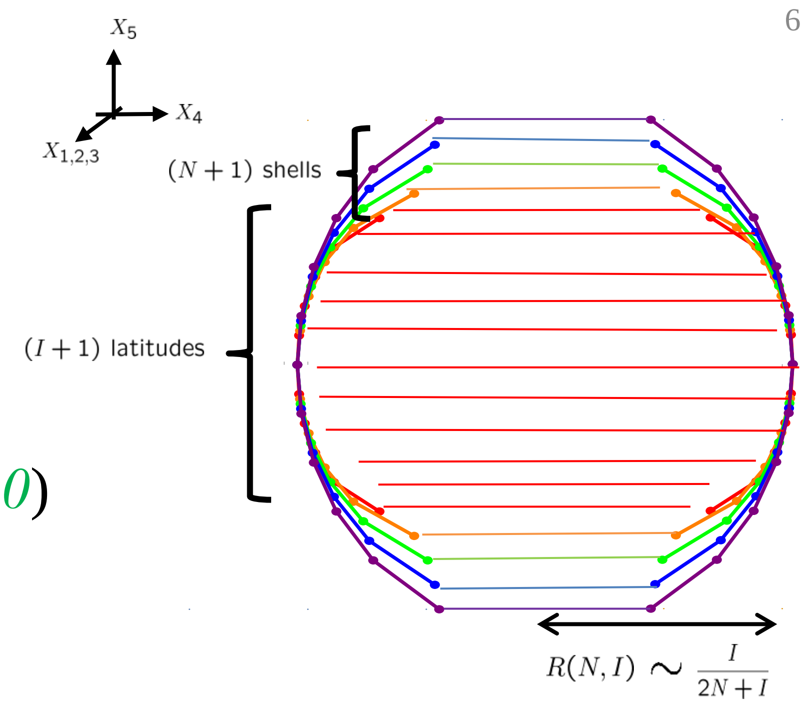
Emergent Quantum Matrix Geometry II

Quantum Nambu geometry

Higher Landau Level projection

➡ nested 4D fuzzy sphere

(*KH '23, '20*)



Quantum Nambu structure

$$[X_a, X_b, X_c, X_d] = 4!\epsilon_{abcde}X_e$$

➡ New solution of Yang-Mills matrix model with Chern-Simons term

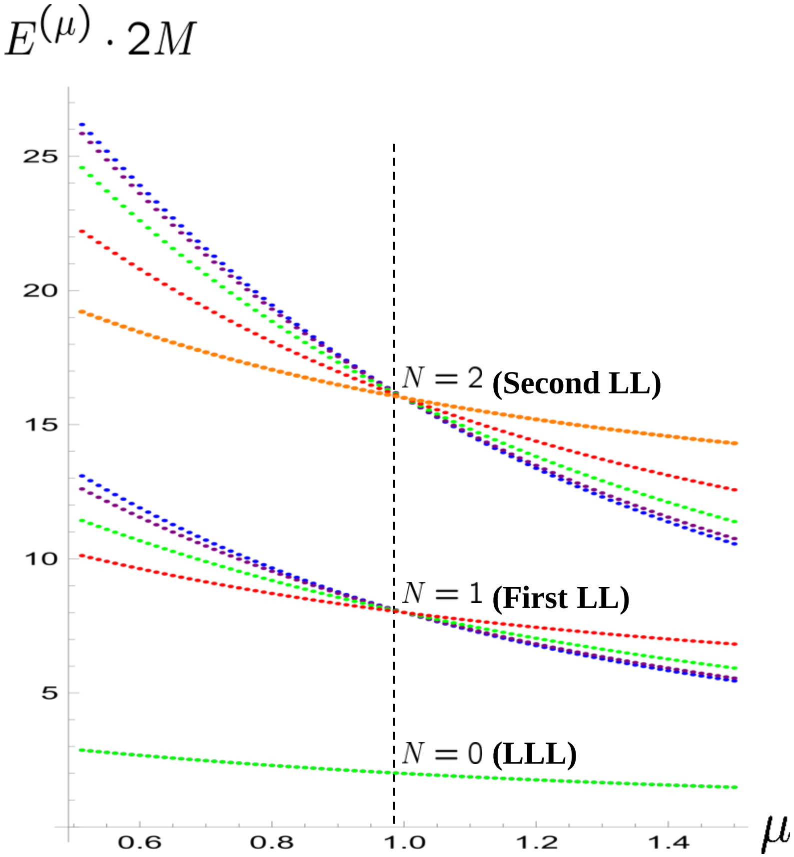
$$[[X_a, X_b], X_b] = \epsilon_{abcde}X_bX_cX_dX_e$$

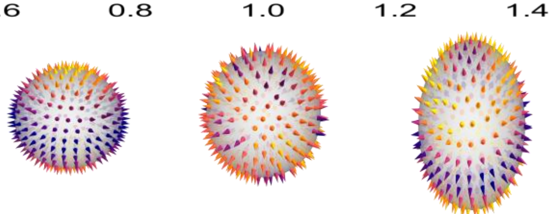
Emergent Quantum Matrix Geometry III

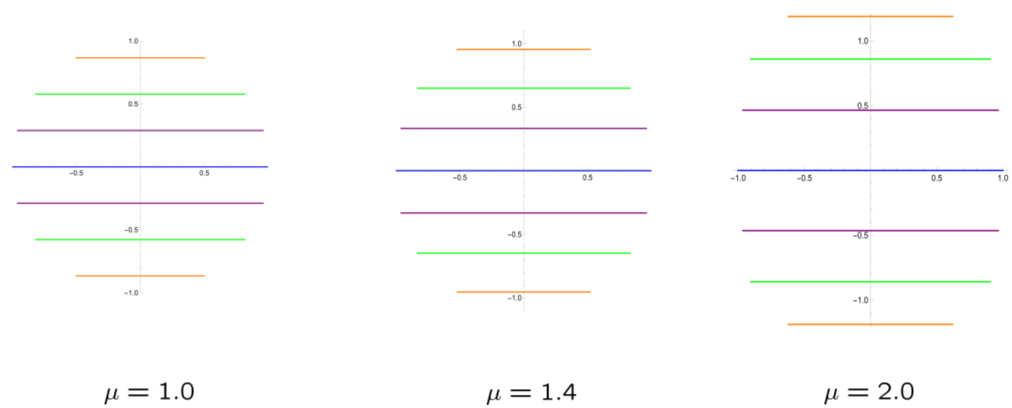
Deformed fuzzy geometry

Spectral flow ⟹ ``Deformed'' quantum states

⟹ Deformed fuzzy space (*KH '25*)







(see also *Matsuura, Tsuchiya '20*)

Matrix geometry and Quantum states

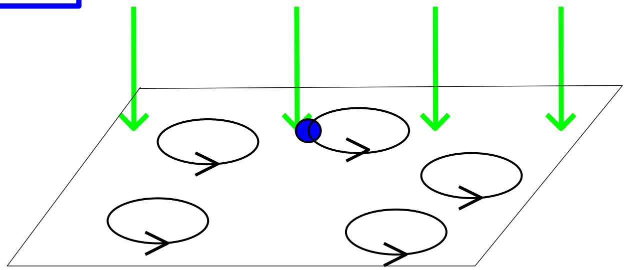
Matrix model

(Matrix) Geometry (vacuum) ⟹ Fluctuations (field theory) ⟹ Quantum states

True quantum vac.
(*Nishimura et al*.)

Hopefully,
not just conceptual, but
practically useful !

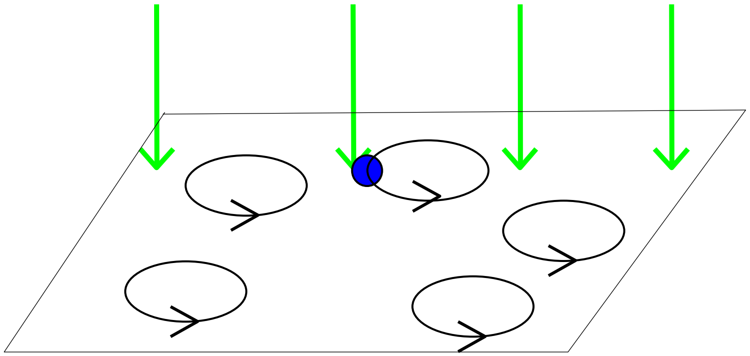
Landau model



Level projection ⟹ Quantum states ⟹ (Matrix) Geometry

The set-up

Landau ('30)



NC plane

$$[X,Y]=i\ell^2$$

$$\ell = 1/\sqrt{B}$$

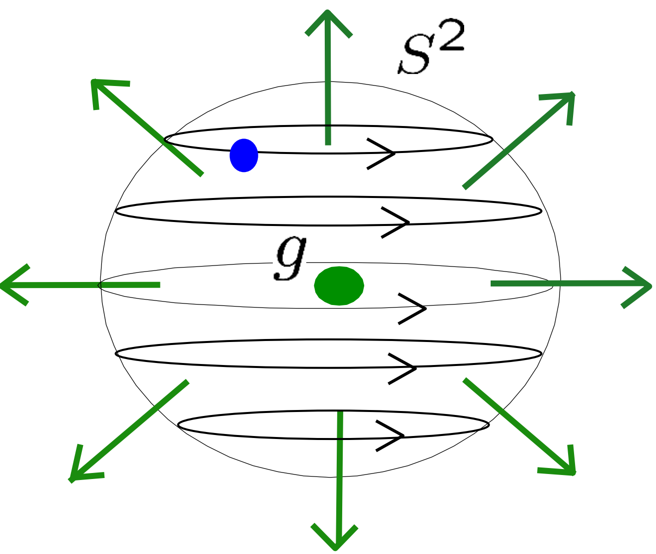
Non-compact space

$$A=\infty$$

Degeneracy $= \infty$

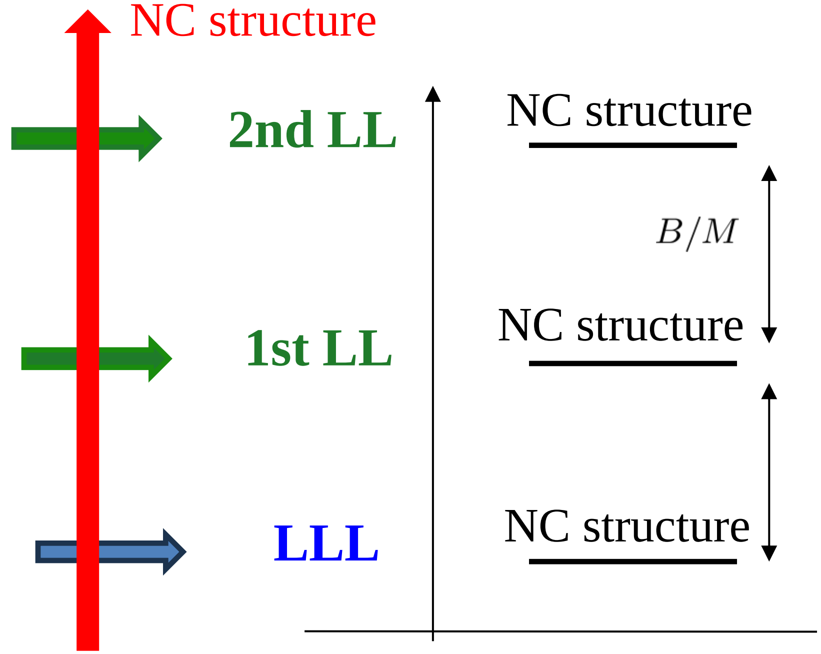
$$A/(2\pi\ell^2)$$

$g,\ R \to \infty \quad (g/R^2\ :\ \text{fixed})$



Level projection

BT quantization



Compact space

Degeneracy : finite

(Higher D. : non-trivial structure in higher LLs)

The Hopf maps and fuzzy spheres

The first Hopf map $\quad S^3 \overset{S^1}{\rightarrow} S^2 \quad$ (*Hopf '31*) $\qquad$ (*Review KH '10*)

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad \rightarrow \quad x_i = \begin{pmatrix}\phi_1^*\\ \phi_2^*\end{pmatrix}^t \sigma_i \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

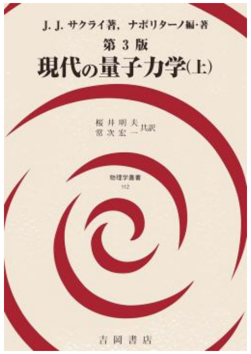
$$\phi_1^*\phi_1 + \phi_2^*\phi_2 = 1 \qquad\qquad x_i x_i = (\phi_1^*\phi_1 + \phi_2^*\phi_2)^2 = 1$$

The fuzzy sphere $\quad$ From quantum state to geometry

$$[\hat{\phi}_1, \hat{\phi}_1^\dagger] = [\hat{\phi}_2, \hat{\phi}_2^\dagger] = 1 \quad \rightarrow \quad X_i = \begin{pmatrix}\hat{\phi}_1^\dagger\\ \hat{\phi}_2^\dagger\end{pmatrix}^t \sigma_i \begin{pmatrix}\hat{\phi}_1\\ \hat{\phi}_2\end{pmatrix}$$

$$[\hat{\phi}_1, \hat{\phi}_2] = [\hat{\phi}_1, \hat{\phi}_2^\dagger] = 0$$

(Two independent quantum oscillators) $\qquad [X_i, X_j] = i\epsilon_{ijk}X_k$



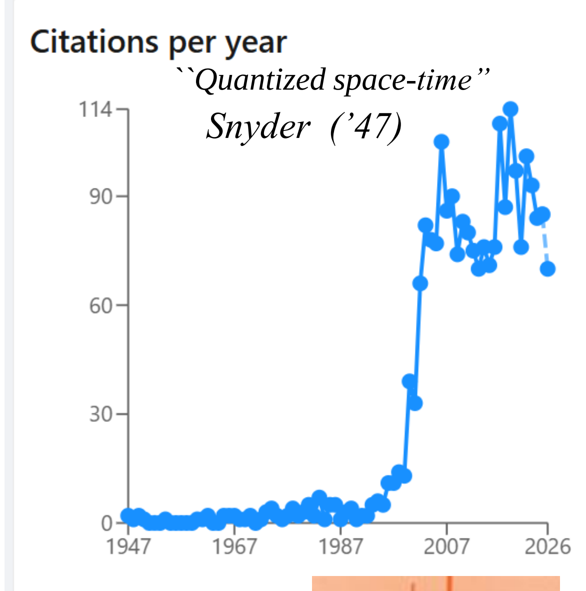
The second Hopf map, third Hopf map… (*Hopf '31*) $\Rightarrow$ Higher D fuzzy spheres (*Grosse Klimcik Presnajder '95,...Kimura KH '04...*)

The graded Hopf maps (*Landi, Marmo '87.. KH '11*) $\Rightarrow$ Fuzzy superspheres (*Grosse Klimcik Presnajder '95,.., KH '11,'26*)

The non-compact Hopf maps (*KH '10*) $\Rightarrow$ Higher D fuzzy hyperboloids (*KH '12 ..*)

Non-Hermitian topological phase (Esaki, Sato, KH, Kohmoto '11), squeezed state (KH '19)..

Fuzzy sphere



Matrix coordinates *Hoppe ('82) Madore ('92)*

$$[X_i, X_j] = i\alpha\epsilon_{ijk}X_k$$

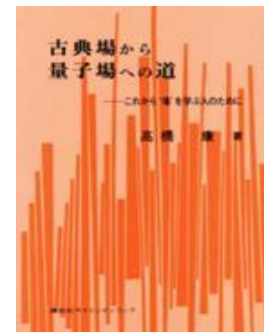
$$\sum_{i=1}^{3} X_i X_i = \alpha^2 l(l+1)1_{2l+1} \qquad (X_i = \alpha S_i^{(l)})$$

Discrete but continuous rotational symmetry!

$$X_i \quad \rightarrow \quad R_{ij}(\theta,\phi)X_j$$

[cf.] *Snyder space-time ('47)*

「…理論が相対性理論と矛盾せず、…
彼の提出した理論に続くものがいない」
（第12版1993年）



Matrix Model Solution

Myers ('99), Iso et al ('01), Kimura ('01, '02), Gohara Sako ('25)..

$$S = \mathrm{tr}[X_i, X_j]^2 + X_i X_i \qquad S = \mathrm{tr}[X_i, X_j]^2 + \epsilon_{ijk}X_i X_j X_k$$

Numerical simulations of 3D Ising model (Fuzzy sphere regularization)

Zhu et al. ['22~] …

Annual Review of Condensed Matter Physics

A Fuzzy Sphere Journey in Critical Phenomena

Yin-Chen He[1] and W. Zhu[2]

https://scgp.stonybrook.edu/archives/41863

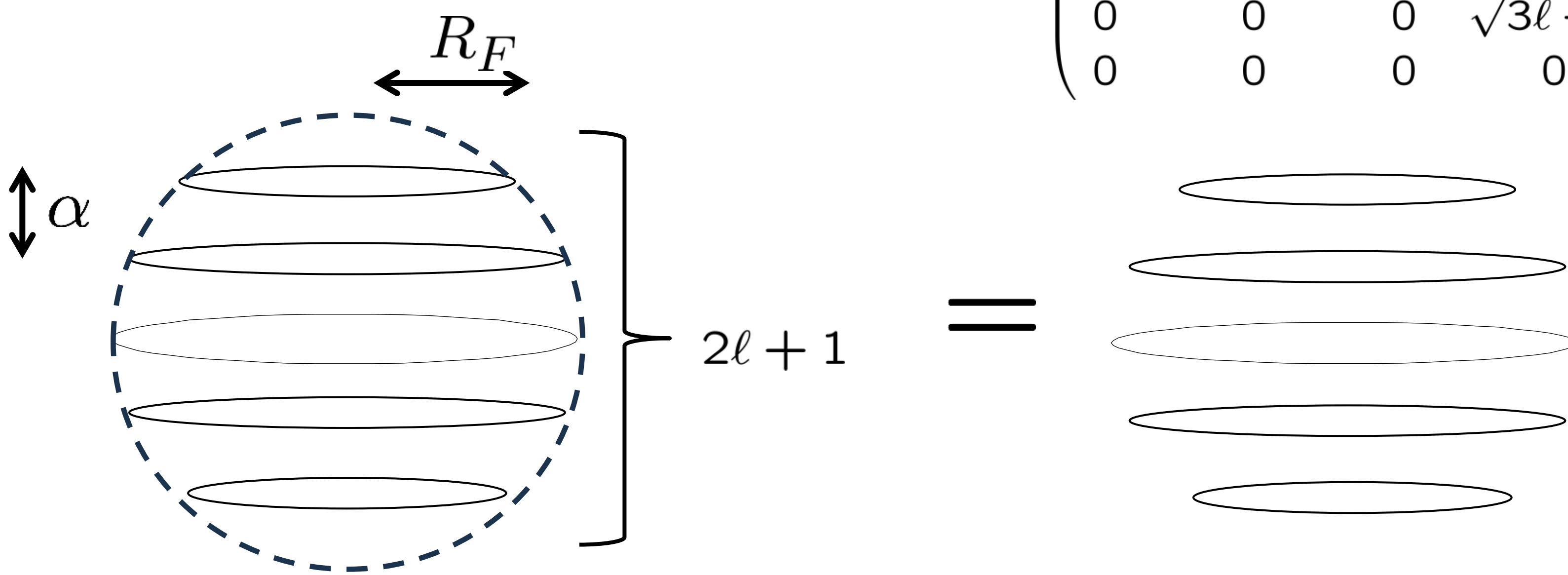
https://indico.math.cnrs.fr/event/13856/

Intuitive picture of the fuzzy sphere

$$X_3 = \alpha \begin{pmatrix} \ell & 0 & 0 & 0 & 0 \\ 0 & \ell-1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & -(\ell-1) & 0 \\ 0 & 0 & 0 & 0 & -\ell \end{pmatrix}$$

$$R_F = \sqrt{{X_1}^2 + {X_2}^2} = \sqrt{\boldsymbol{X}^2 - {X_3}^2}$$

$$= \alpha \begin{pmatrix} \sqrt{\ell} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3\ell+1} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3\ell+1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\ell} \end{pmatrix}$$

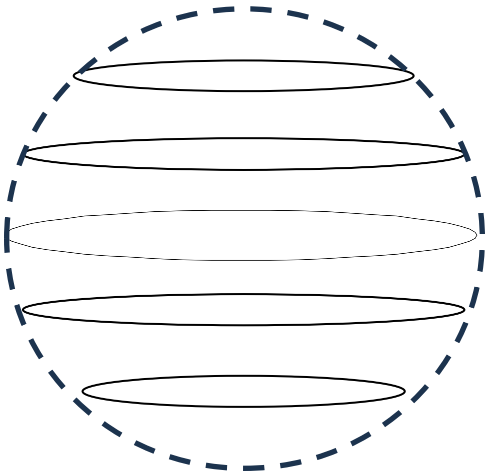


Fuzzy sphere

Quantum states (irrep)

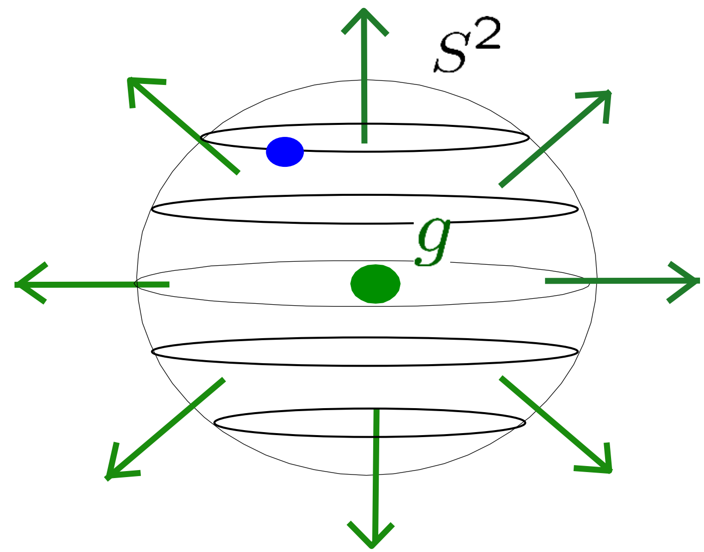
Quantum states => Fuzzy geometry

Physical model: Haldane's sphere



Fuzzy sphere

$\Longleftrightarrow$



Haldane's sphere

Haldane ('83)

Monopole *Dirac ('31)*

$$A_i^{(g)} = -g\frac{1}{r(r+x_3)}\epsilon_{ij3}x_j \qquad F_{ij}^{(g)} = g\frac{1}{r^3}\epsilon_{ijk}x_k$$

$$(g = 0, 1/2, 1, \cdots)$$

Landau model on a sphere

Landau Hamiltonian on a two-sphere *Wu, Yang ('76) Haldane ('83) KH ('16)*

$$H^{(g)} = -\frac{1}{2M}\sum_{i=1}^{3} {D_i}^2\Big|_{r=1} = \frac{1}{2M}\sum_{i=1}^{3} {\Lambda_i}^2$$

$$(\Lambda_i = -i\epsilon_{ijk}x_j(\partial_k + iA_k))$$

SU(2) angular momentum

$$\boxed{L_i = \Lambda_i + g\frac{1}{r}x_i}$$

SU(2) index

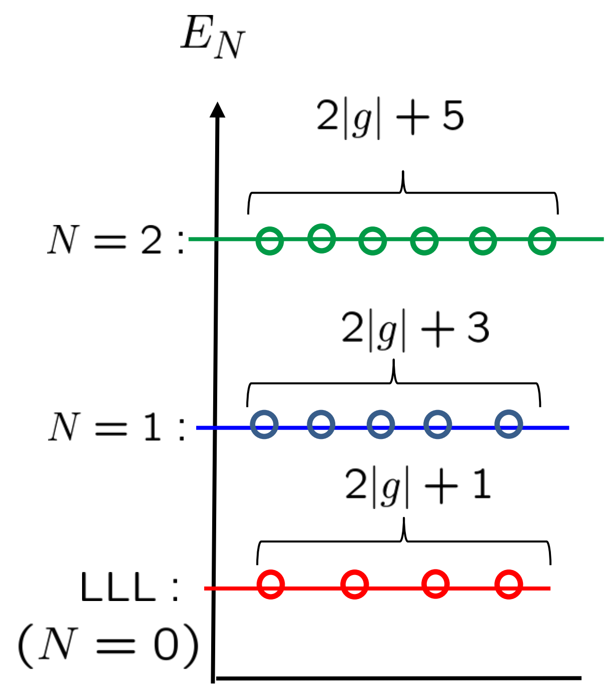
$$\boxed{\ell = N + |g|}$$

$$(N = 0, 1, 2, \cdots)$$

Landau levels

$$E_N = \frac{1}{2M}(2|g|(N+\frac{1}{2}) + N(N+1))$$

$$d_N = 2\ell + 1 = 2|g| + 2N + 1$$



Eigenstates : monopole harmonics

$$\boxed{Y^g_{l,m}(\theta,\phi)} \quad (m = \ell, \ell-1, \cdots, -\ell)$$

NCG

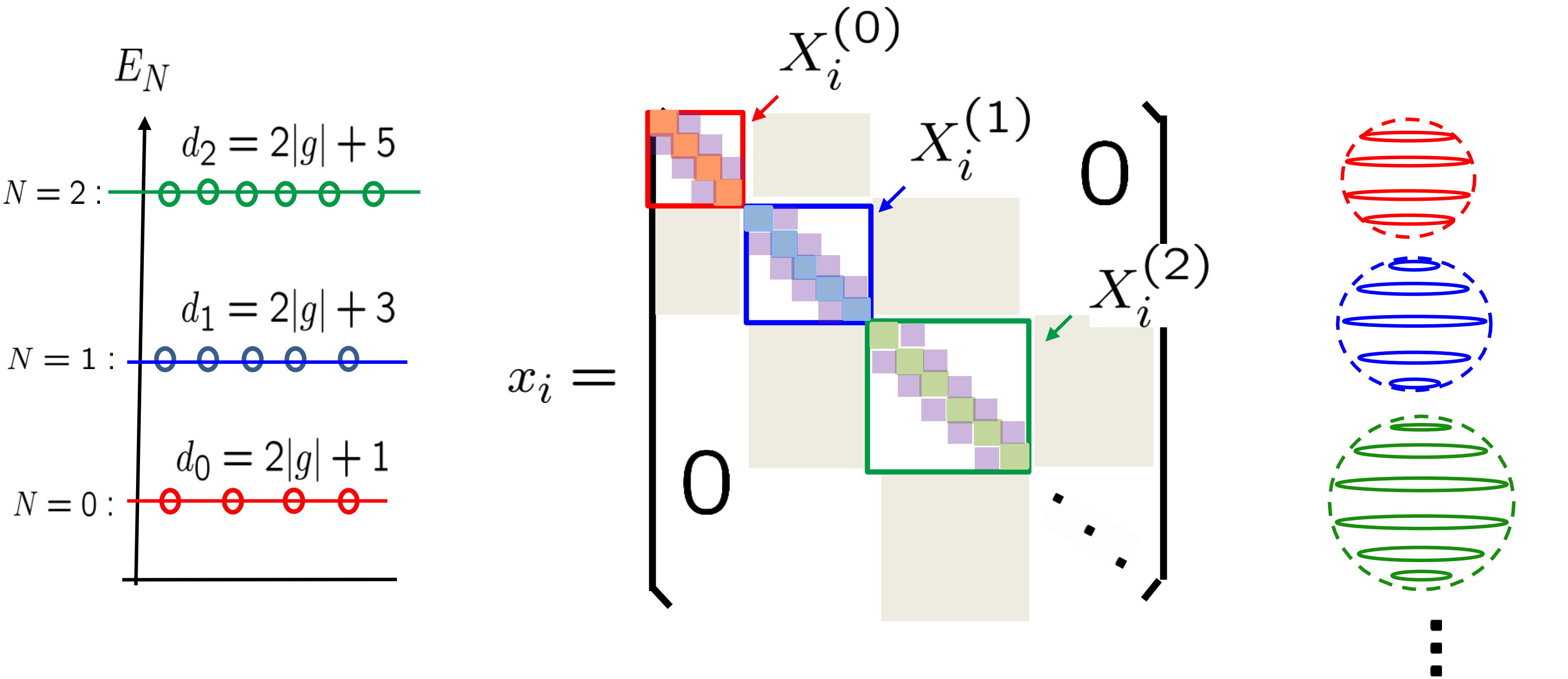
$$[L_i, L_j] = i\epsilon_{ijk}L_k \xrightarrow{\Lambda_i \simeq 0} \boxed{[X_i, X_j] \simeq i\frac{r}{g}\epsilon_{ijk}X_k}$$

Emergence of fuzzy spheres from LLs

How can we precisely evaluate the LL geometries including higher ones?

KH [’*16*]

$$(X_i)_{m_1,m_2} \underset{\ell = N+|g|}{=} \langle Y^g_{l,m_1}|x_i|Y^g_{l,m_2}\rangle|_{S^2} = \frac{g}{l(l+1)}(S^{(l)}_i)_{m_1,m_2}$$



Matrix coordinates

Fuzzy two-spheres

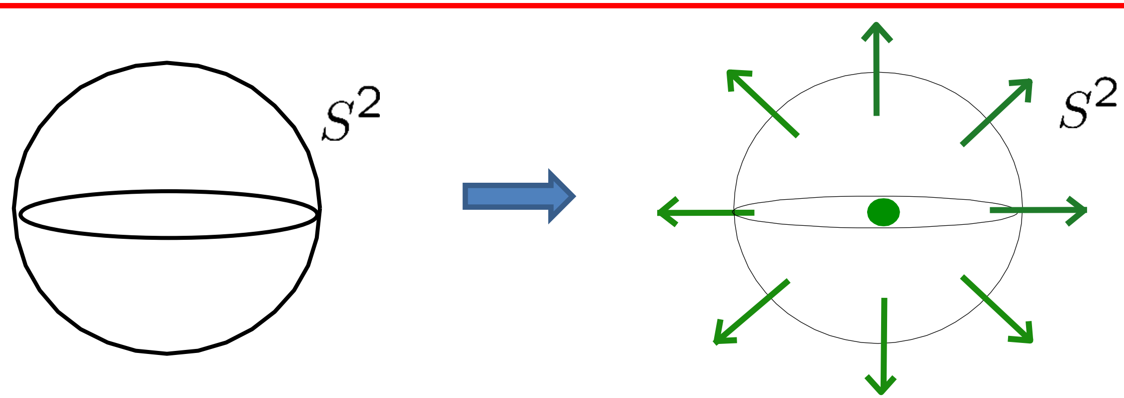
Behind the scene

Reinterpretation S^2 ⇒ S_F^2

- Global sym. of manifold, SO(3)
- Points on the manifold (= infinite dof)
  : a closed set under the SO(3) group action
- Stabilizer group SO(2)
  = sym. that does not change a point

- Global sym. group of the system, SU(2)
- Irreps. (= finite dof) of SU(2)
  : a closed set under the SU(2) group action
- Gauge group U(1)
  = sym. that does not change a physical state

$$\boxed{S^2 \simeq SO(3)/SO(2) \Rightarrow \text{QM with SU(2) global sym. and U(1) gauge sym.}}$$



(Magnetic field is *not* of primary importance; consequence of gauge sym.)

General prescription

$$\mathcal{M} \simeq G/H \Rightarrow \mathcal{M}_F$$

Just replace SU(2) → G, and U(1) → H.

KH ['23]

$\mathcal{M} \simeq G/H$ ⇒ QM with gauge symmetry H on the manifold $\mathcal{M}$

$$H = -\frac{1}{2M}\sum_i {D_i}^2\Big|_{\mathcal{M}}$$

⇒ Solve the eigenvalue problem ⇒ Matrix coordinates $\mathcal{M}_F$

$$|\psi_\alpha^{(N)}\rangle \qquad (X_a^{(N)})_{\alpha\beta} = \langle\psi_\alpha^{(N)}|x_a|\psi_\beta^{(N)}\rangle$$

(Unique feature)

stabilizer group → gauge group

External space residual symmetry → Internal space gauge symmetry

Important Points

Quantum geometry is not postulated a priori,
but naturally emerges in context of physics.

- The original system is totally physical,
  and its mathematical background is a consistent Hilbert space.

  → Hopefully, no need to worry about mathematical inconsistency

- Following the prescription, we can automatically construct the quantum matrix geometry corresponding to G/H.

  → Not restricted to symplectic manifolds; odd D is OK

- Not restricted to commutator formalism

  → Chance for exotic geometry such as Nambu geometry to appear

The monopole harmonics

$$Y^{g}_{l,m}(\theta,\phi)$$

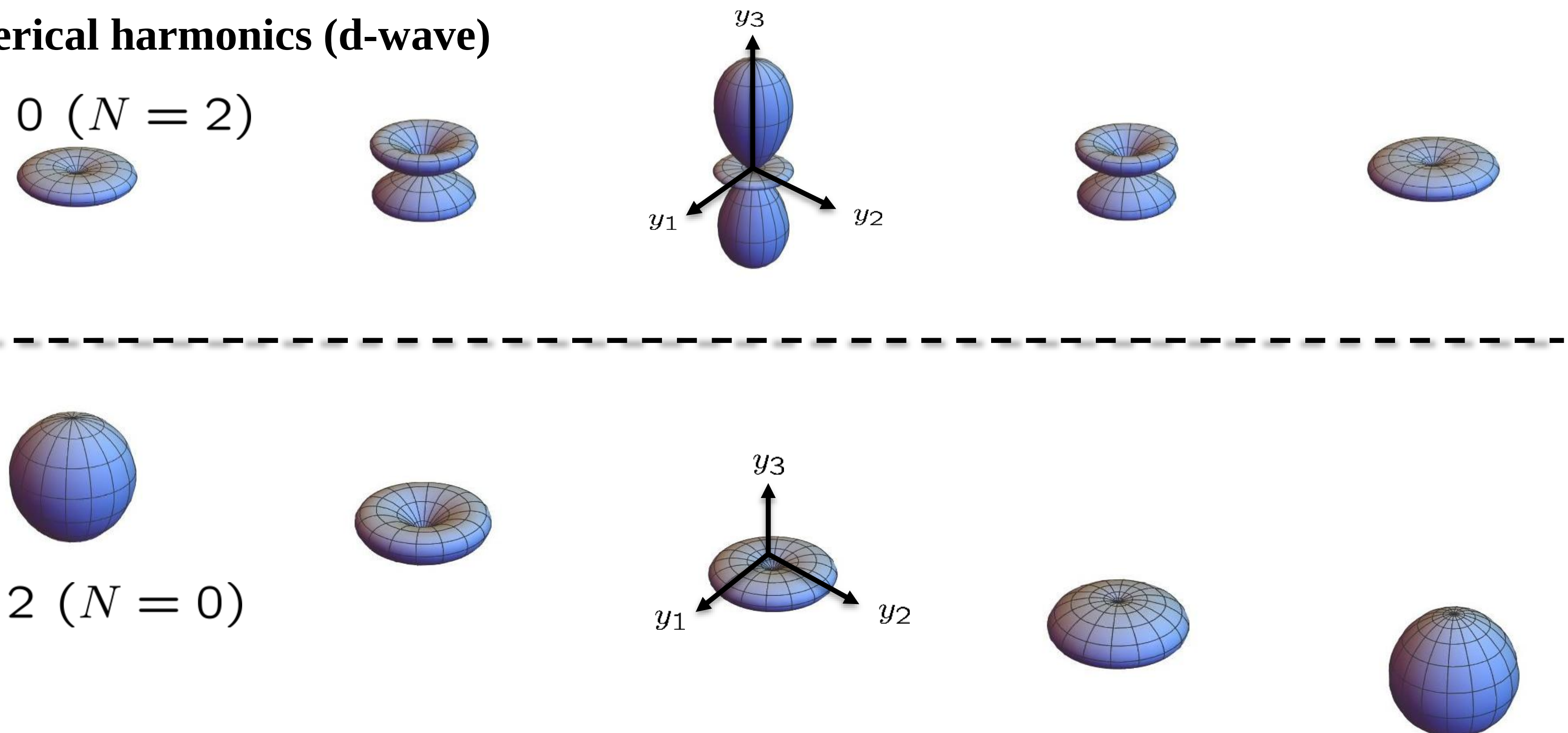
$$\ell = N + |g|$$

$$m = l, l-1, l-2, \cdots, -l$$
$$g = l, l-1, l-2, \cdots, -l$$

$$l = 2$$

Spherical harmonics (d-wave)

$g = 0\ (N = 2)$

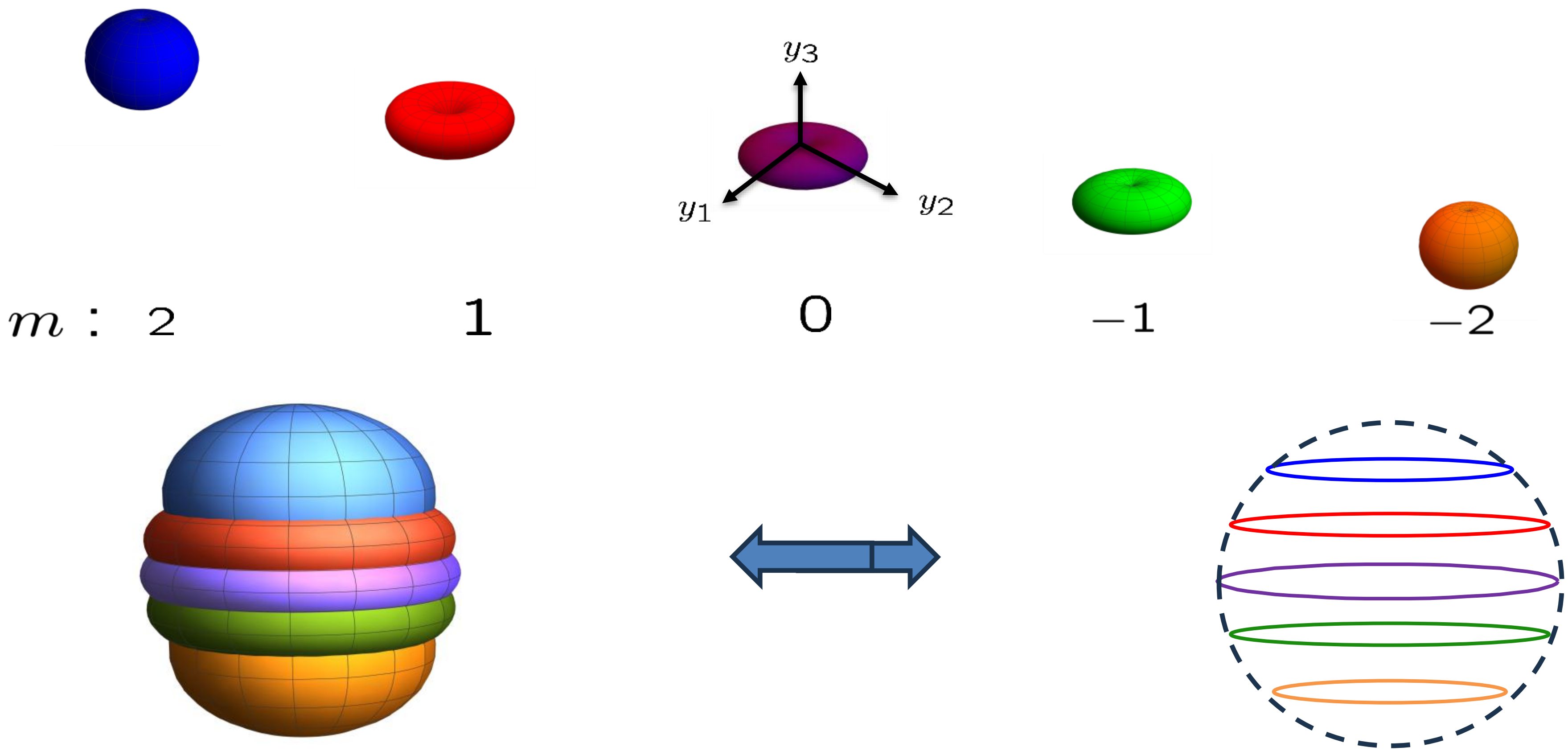


$m:\ \ 2 \qquad 1 \qquad 0 \qquad -1 \qquad -2$

LLL eigenstates

Fuzzy sphere from Quantum states

LLL eigenstates $(l = 2)$



Simple understanding of fuzzy sphere geo. from quantum states !

Landau level eigenstates = SU(2) irreps.= ``Points'' of fuzzy S2

⇨ SU(2)-symmetric NC space = SU(2) irreps.

Irreps. as Quantum Geometry

$$\langle Y^g_{l,m_1} | x_i | Y^g_{l,m_2} \rangle \;\propto\; (S_i^{(l)})_{m_1,m_2}$$ (obvious, from Wigner-Eckart theorem)

Irreps. is important

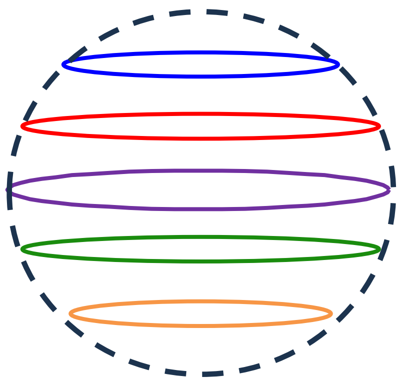
Let's forget about the Landau model. *Concentrate on the quantum states.*

SU(2) rotation

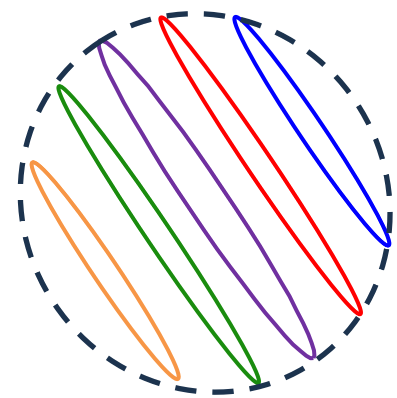
$$\{Y^g_{l,m}(\theta,\phi)\} \qquad \{Y^g_{l,m}(\theta,\phi)\}$$

A set of a finite # of quantum states

SU(2) ``invariant''



SU(2) rotation



A set of a finite # of quantum elements

SU(2) symmetry

Discrete (Finite number) but continuous rotational symmetry !

Quantum states (Irreps.)→ Quantum Geometry

Monopole harmonics and spin-coherent state

$l = 1/2$ → 2-fold degeneracy

$(l = N + |g|)$ $\quad g = \pm 1/2, N = 0$

$$
\begin{array}{c c}
 & \begin{array}{cc} m = +1/2 & m = -1/2 \end{array} \\
\begin{array}{c} g = +1/2 \\ g = -1/2 \end{array} & \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\phi} \\ -\sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}
\end{array}
= e^{i\frac{\chi}{2}\sigma_z} e^{i\frac{\theta}{2}\sigma_y} e^{i\frac{\phi}{2}\sigma_z}\Big|_{\chi=-\phi}
$$

S=1/2 spin-coherent state $\qquad$ S=-1/2 spin-coherent state

$$\frac{1}{2}\boldsymbol{\sigma} \cdot \widehat{\boldsymbol{x}}(\theta, \phi)\psi_S = S\psi_S$$

Row → degenerate LL eigenstates (→ from classical geo to matrix geo.)

Column → (spin-)coherent state (→ from matrix geo. to classical geo.)

→ *Ishiki ('15)*

D-matrix as a unified description

$$Y^g_{l,m}(\theta,\phi) \qquad m = l, l-1, l-2, \cdots, -l \qquad g = l, l-1, l-2, \cdots, -l$$

Wu Yang('77)   Dray('86)   KH (`16, '23)

l : fixed $\quad (l = N + |g|)$

$2l+1$, $2l+1$

L_- →, ← L_+

$g \backslash m$	l	$l-1$	$l-2$	$\cdots$	m	$\cdots$	$-l+1$	$-l$
l	LLL eigenstates							
$l-1$	1^{st} LL eigenstates							
$l-2$								
$\vdots$								
g					$Y^g_{l,m}$			
$\vdots$								
$-l+1$								
$-l$								

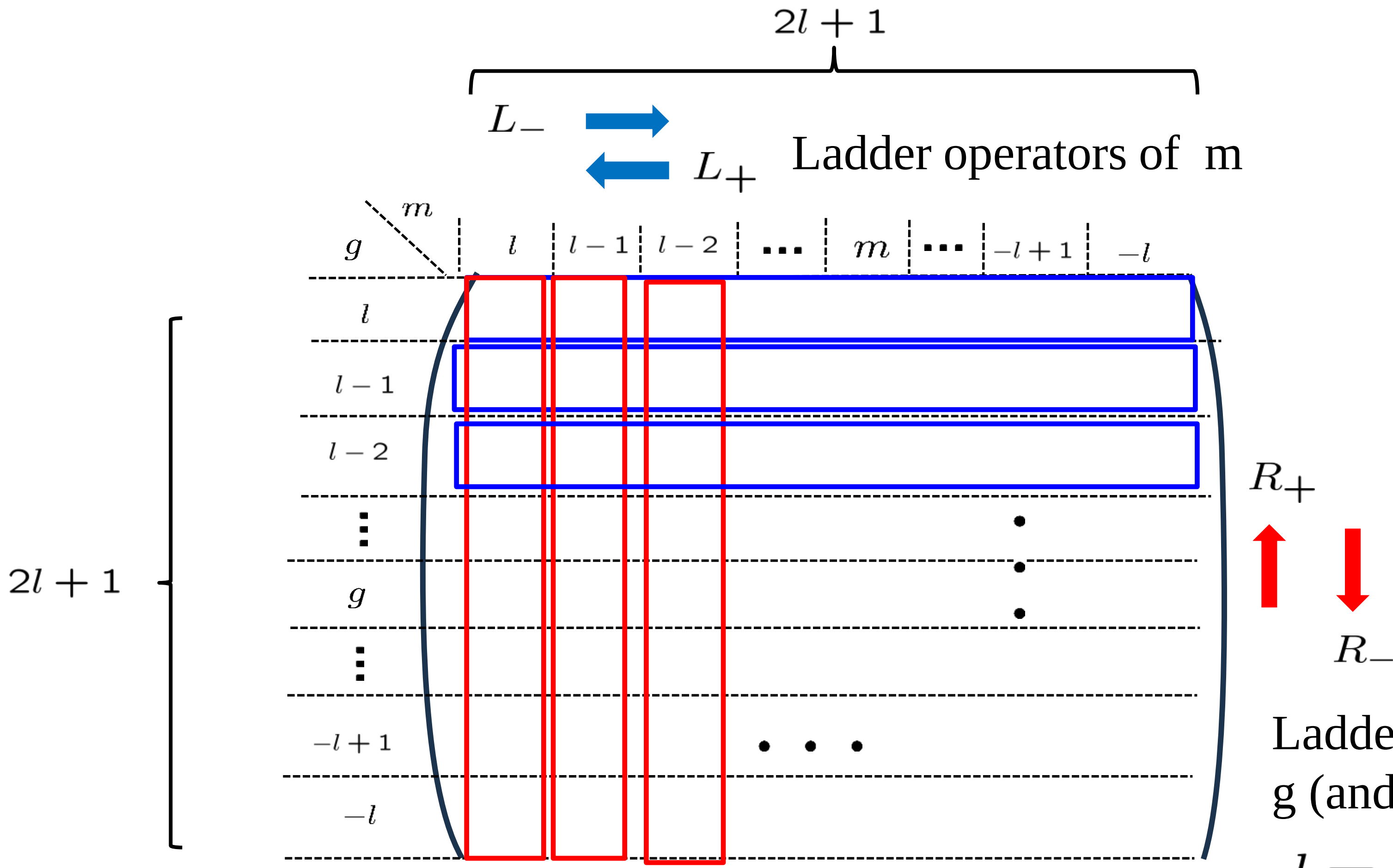
(D matrix)

$$= e^{i\chi S_z^{(l)}} e^{i\theta S_y^{(l)}} e^{i\phi S_z^{(l)}}\Big|_{\chi=-\phi}$$

(= Euler angle rep. of SU(2))

** General feature of the Landau models on G/H*

Two kinds of Ladder operators for D-matrix



Ladder operators of g (and N)

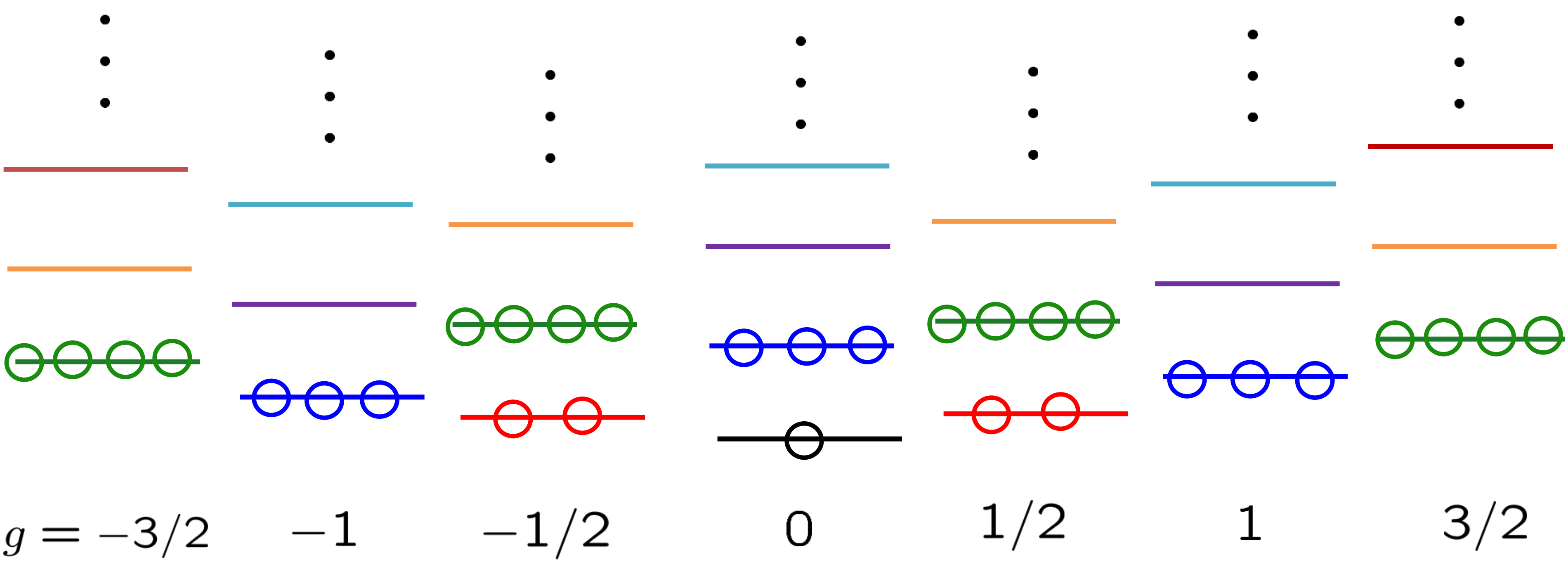
$$l = N + |g|$$

$$(\Delta N = -\Delta |g|)$$

$\mathcal{H} = \oplus_{l=0}^{\infty} \mathcal{H}^{(l)}$ Giant Hilbert space of a set of Hamiltonians $\{H^{(g)}\}$

Howe duality from extended Landau levels

$$H^{(g)} = -\frac{1}{2M}\sum_{i=1}^{3} {D_i}^2\Big|_{r=1} \qquad l = N + |g| \qquad \mathrm{dim} = 2l+1$$



The degeneracy is exactly equal to the number of the corresponding LLs.

Howe duality is a hidden structure of a whole set of the Landau models.

(We cannot discover the Howe dual structure only by analyzing a single system.)

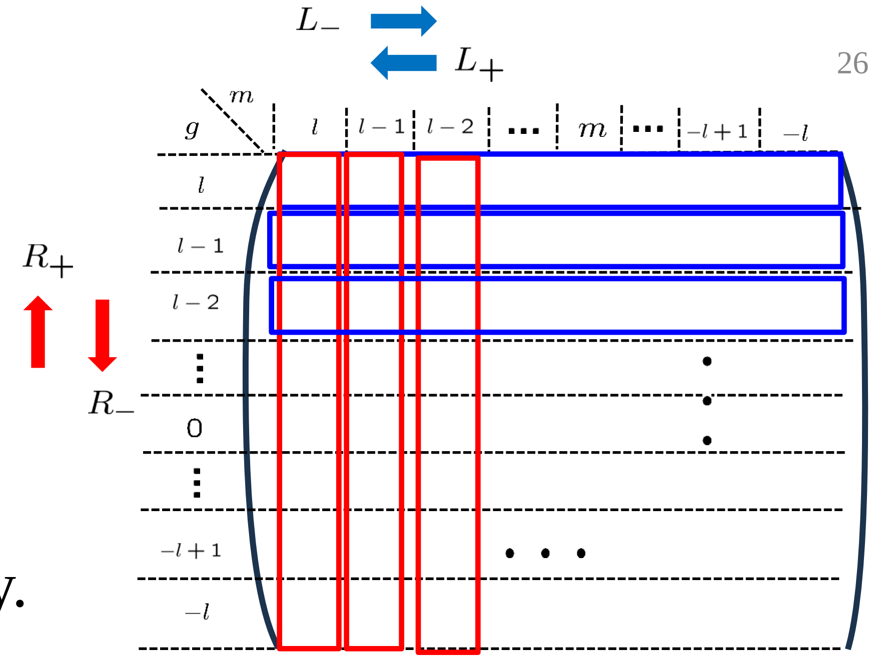
Howe correspondence

- Dual pair $(SU(2)_L,\ SU(2)_R)$

$$[L_i, L_j] = i\epsilon_{ijk}L_k \qquad [L_i, R_j] = 0$$

$$[R_i, R_j] = i\epsilon_{ijk}R_k$$

Karabali Nair ['02] *Greiter* ['11]



SU(2)R: spectrum generating algebra,
not a symmetry.

Momentum operators
(Newman-Penrose' edth operators *['66]*)

$$R_\pm \ \rightarrow\ \eth_\pm^{(g)} := D_\theta \pm i\frac{1}{\sin\theta}D_\phi$$

KH ['16]

$\Rightarrow$

R_i exactly coincide with the covariant derivatives ∇_i given by
Hanada, Kawai, Kimura (2005)

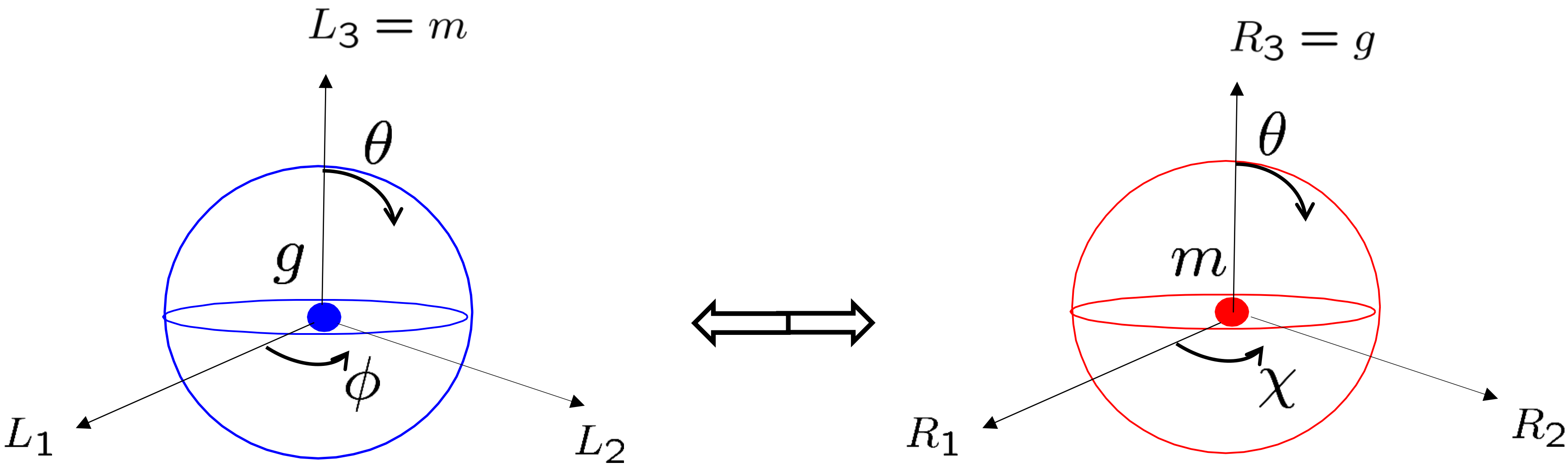
- Howe-correspondence

$$\mathcal{H} = \oplus_{l=0}^{\infty}(V^{(l)}_{SU(2)_L} \otimes V^{(l)}_{SU(2)_R})$$

: Howe duality

Dual Landau model

$$L_iL_i = R_iR_i = l(l+1)$$



$$X_i = L_i \qquad \nabla_i = R_i$$

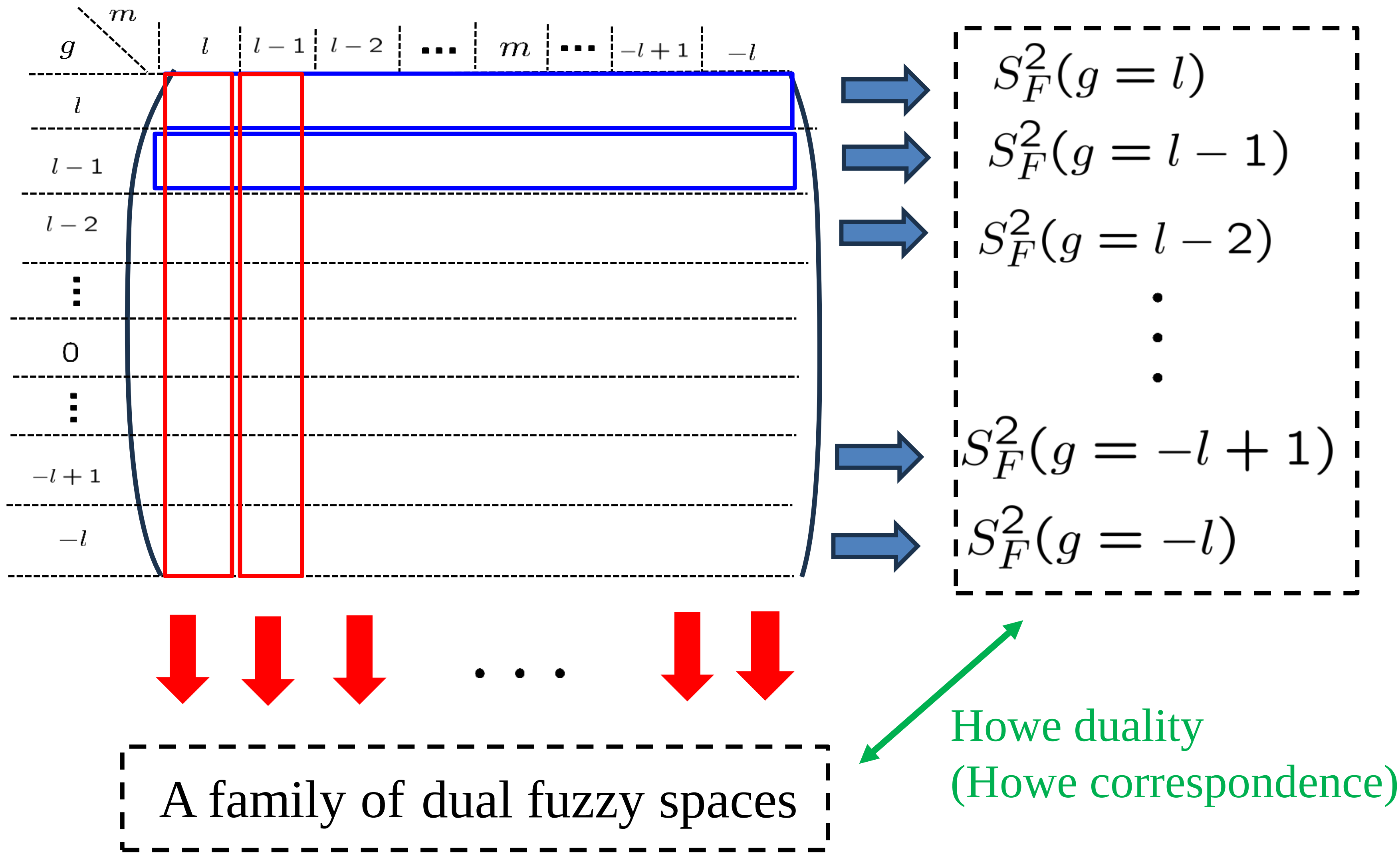
$$D_{l,g,m}(\chi,\theta,\phi) = (e^{i\chi S_z^{(l)}} e^{i\theta S_y^{(l)}} e^{i\phi S_z^{(l)}})_{g,m} = e^{ig\chi} d_{l,g,m}(\theta) e^{im\phi}$$

Quantization of magnetic quantum number → topological charge in the dual system

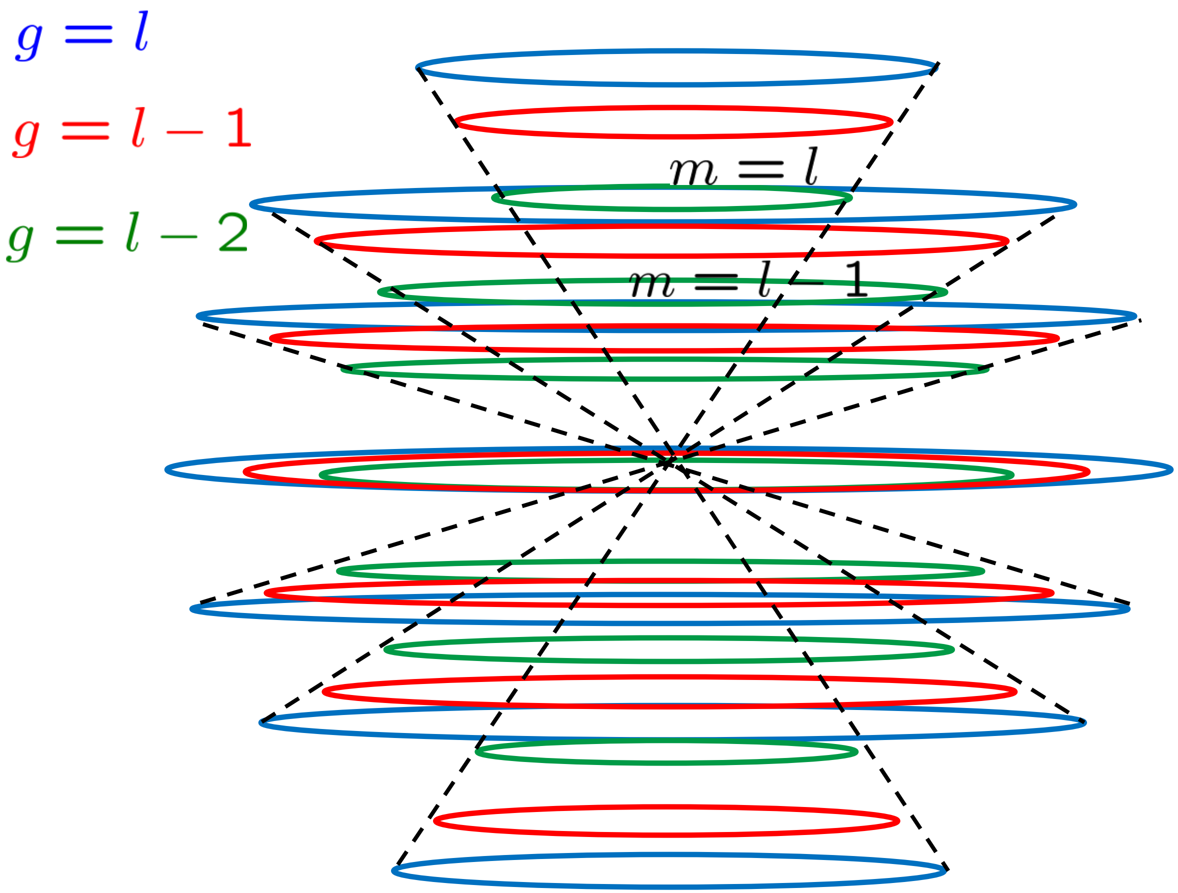
Howe duality

$l = N + |g|$

A family of fuzzy spheres

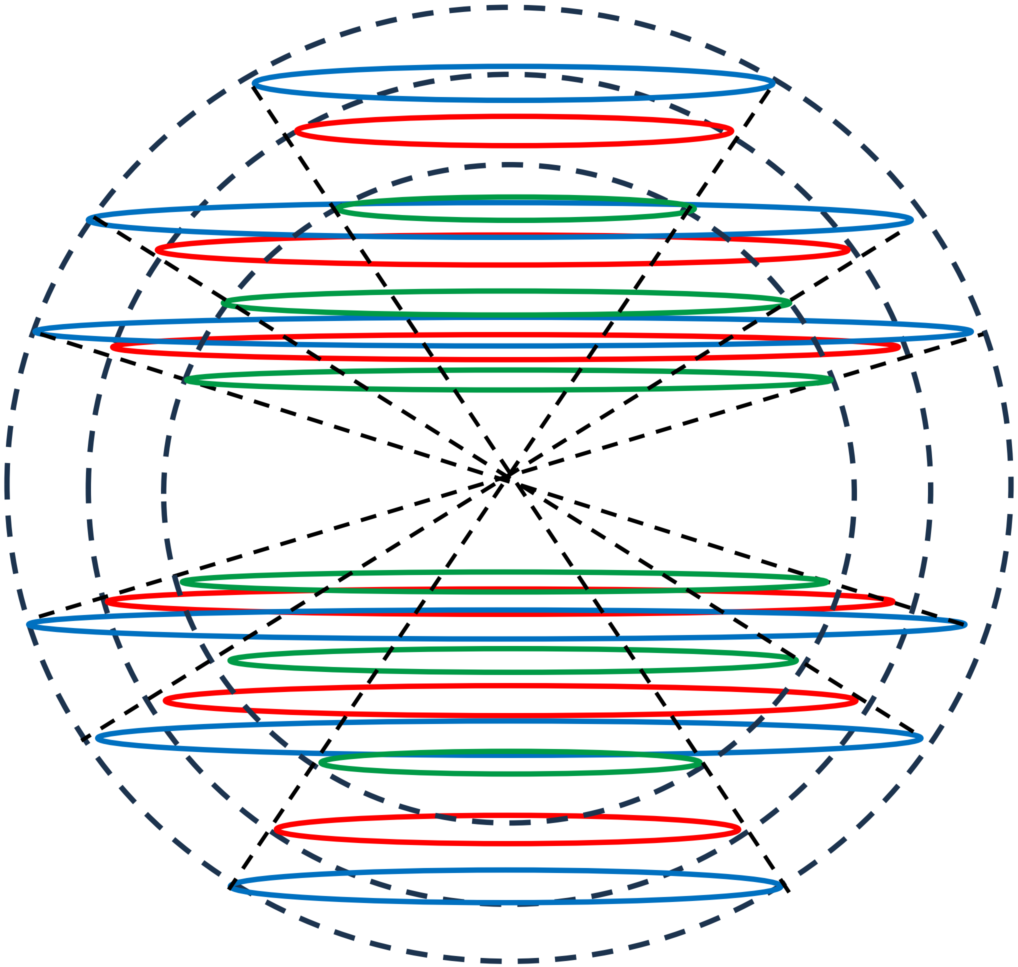


Emergence of dual fuzzy geometry



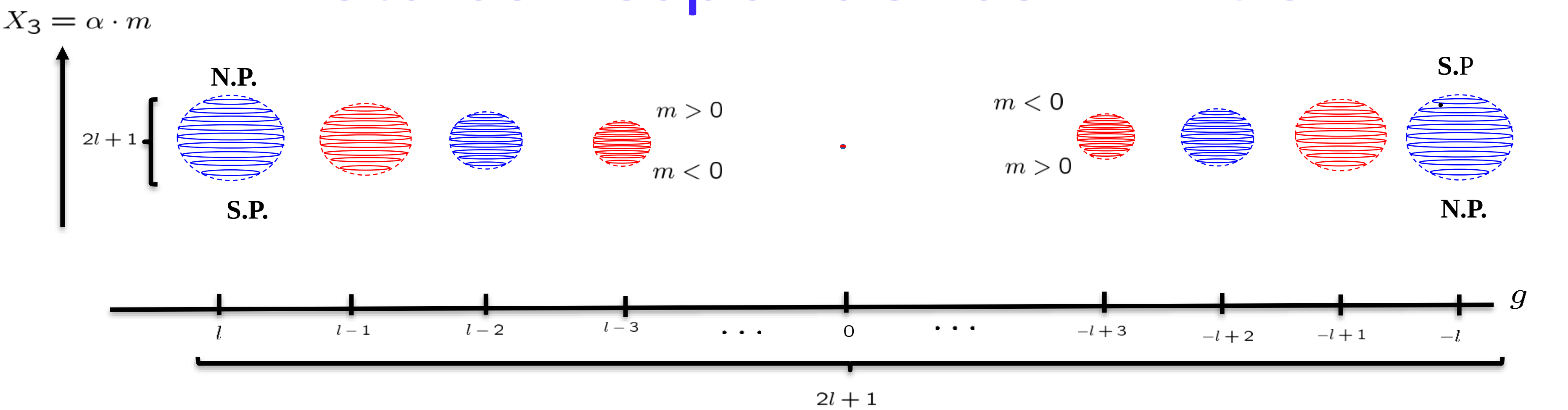
Dual geometry emerges once the whole structure is identified.

Intuitive understanding of the duality



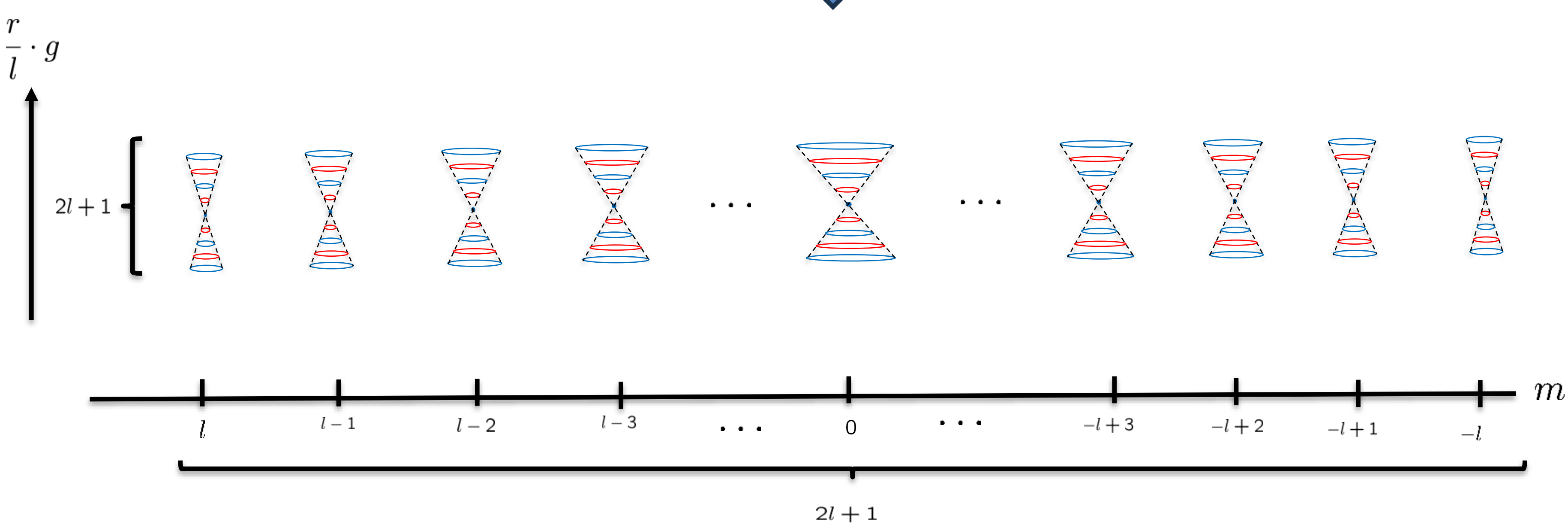
Two faces of single object

Theta correspondence in NCG



$$X_i = \frac{g}{l(l+1)} S_i^{(l)}$$

Howe correspondence



Internal and external spaces in Matrix geo.

Fuzzy S4 coordinates

$SO(5)$ $\gamma_{a=1,2,3,4,5}$

$$[\gamma_a, \gamma_b] = i\sigma_{ab}$$

External space *Internal space*

(*Ho-Ramgoolam 2002, Kimura 2002, Kimura-KH 2004*)

common algebraic origin, same matrix size

$$\gamma_a \Rightarrow X_a \qquad \sigma_{ab} \Rightarrow \Sigma_{ab}$$

→ *Higher spin structure from IKKT model*

(*Steinacker 2016,2019,2023*)

Internal space $\simeq$ External space

momentum A_μ *space-time coordinates*

$(\nabla_i \quad \Leftrightarrow \quad X_i)$ (*Hanada-Kawai-Kimura 2005,*

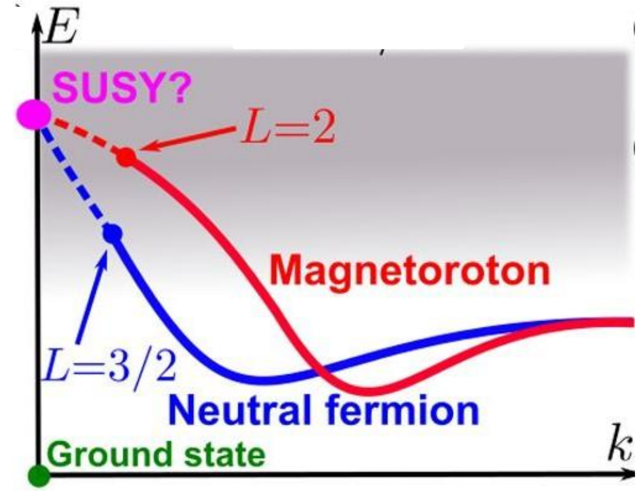
Ho-Kawai-Steinacker 2026)

⇒ Howe duality ?

Renewed interest in SUSY NCG

Moore-Read state (even denominator QHE)

$$\nu = \frac{5}{2}$$



Pu, Balram, et al ('23) PRL 130, 176501

Bulk excitation

- **magnetoroton (L=2)**
- Super partner ?
- **neutral fermion (L=3/2)**

Gromov, Martinec, Ryu ('20), Pu, Balram, Femling, Gromov , Papic ('23), …

supersymmetry in fractional quantum Hall systems -- references in nLab

Exact SUSY of the fuzzy supersphere

supersphere

$$x_i x_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta = R^2$$

$\mathcal{N}=1$ SUSY (UOSp(1|2) group)

$$\begin{cases} \delta_\eta x_i = -\xi_\alpha \frac{1}{2}(\sigma_i)_{\beta\alpha}\theta_\beta \\ \delta_\eta \theta_\alpha = i\xi_\beta \frac{1}{2}(\sigma_2\sigma_i)_{\beta\alpha} x_i \end{cases}$$

Fuzzy supersphere

Grosse Klimcik, Presnajder ('97) *Grosse Reiter* ('98) *Iso Umetsu* ('04)

$$X_i X_i + \epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta \propto \mathbf{1}$$

Exact $\mathcal{N}=1$ SUSY

$$\begin{cases} \delta_\eta X_i = -\xi_\alpha \frac{1}{2}(\sigma_i)_{\beta\alpha}\Theta_\beta \\ \delta_\eta \Theta_\alpha = i\xi_\beta \frac{1}{2}(\sigma_2\sigma_i)_{\beta\alpha} X_i \end{cases}$$

$$[X_i, X_j] = i\epsilon_{ijk}X_k$$

$$[X_i, \Theta_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}\Theta_\beta$$

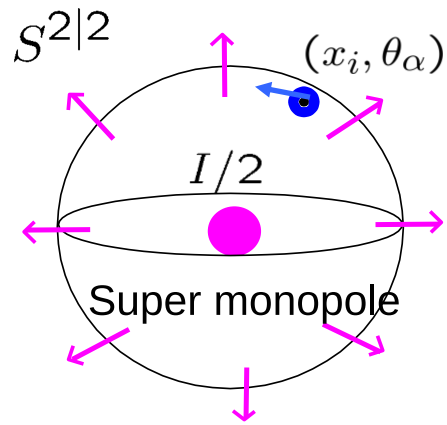
$$\{\Theta_\alpha, \Theta_\beta\} = i\frac{1}{2}(\sigma_2\sigma_i)_{\alpha\beta}X_i.$$

uosp(1|2) algebra

Super Landau model

Super Landau model

Kimura, KH ('04) *KH ('26)*



$$H = \frac{1}{2MR^2}(\Lambda_i\Lambda_i + \epsilon_{\alpha\beta}\Lambda_\alpha\Lambda_\beta)$$

on supersphere $\quad x_i x_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta = R^2$

$$\begin{cases} \Lambda_a = -i\epsilon_{abc}x_b D_c + \frac{1}{2}\theta_\alpha(\sigma_a)_{\alpha\beta}D_\beta \\ \Lambda_\alpha = \frac{1}{2}(\epsilon\sigma_a)_{\alpha\beta}x_a D_\beta - \frac{1}{2}\theta_\beta(\sigma_a)_{\beta\alpha}D_a \end{cases}$$

$$A_i = g\epsilon_{ij3}\frac{x_j}{r(r+x_3)}\left(1 + \frac{r+x_3}{2r^2(r+x_3)}\theta\epsilon\theta\right)$$

$$A_\alpha = g\frac{1}{r^3}i(\sigma_i\epsilon)_{\alpha\beta}x_i\theta_\beta$$

$$E_N = \frac{1}{2MR^2}\left(N\left(N+\frac{1}{2}\right) + \left(2N + \frac{1}{2}|g|\right)\right) \qquad N = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$$

$$d_N = 4l + 1 \qquad l = N + |g|$$

Supermonopole Harmonics

$UOSp(1|2)$ *superspin* $l = 0, 1/2, 1, 3/2, \cdots$ (Ex.) $l = 1$

$\Rightarrow$ $SU(2)$ $l \bigoplus l - \frac{1}{2}$

$$d = \underbrace{(2l+1)}_{\text{Bosonic d.o.f.}} + \underbrace{(2l)}_{\text{Fermionic d.o.f.}} = 4l + 1$$

Bosonic d.o.f. Fermionic d.o.f.

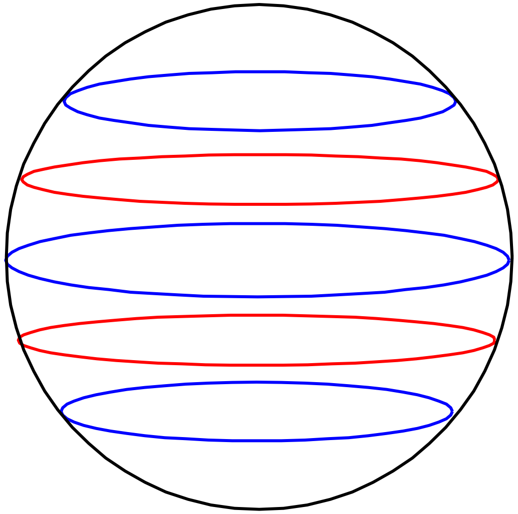
LLL

$$Y_{m_1,m_2} = \phi_1^{m_1}\phi_2^{m_2},$$

$$\mathcal{Y}_{n_1,n_2} = \phi_1^{n_1}\phi_2^{n_2}\eta \qquad (\eta := \phi_1\theta_2 - \phi_2\theta_1)$$

$m_1 + m_2 = 2l \qquad n_1 + n_2 = 2l - 1$

(No complex conjugate variables)



The SUSY system is (almost) doubly degenerate due to the existence of the fermionic d.o.f.

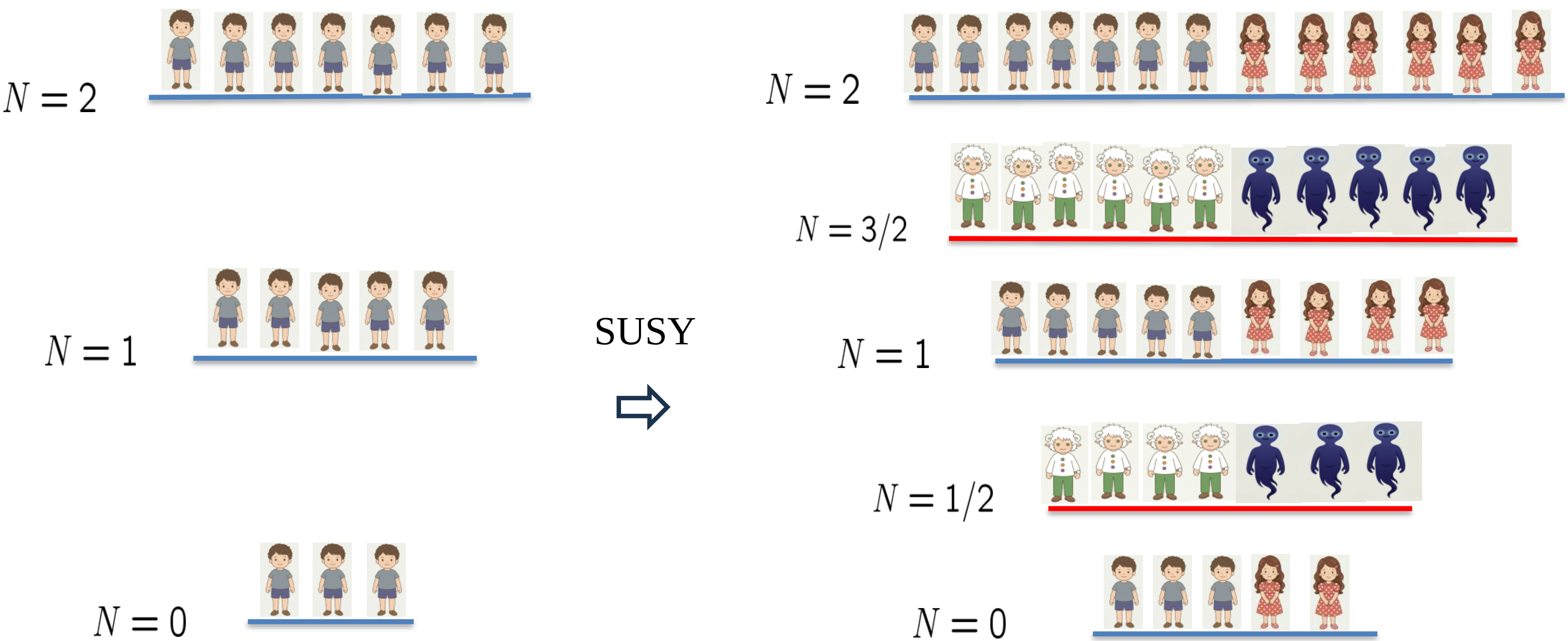
Super Landau Levels and Howe duality

Supermonopole harmonics =

bosonic monopole harmonics + fermionic monopole harmonics

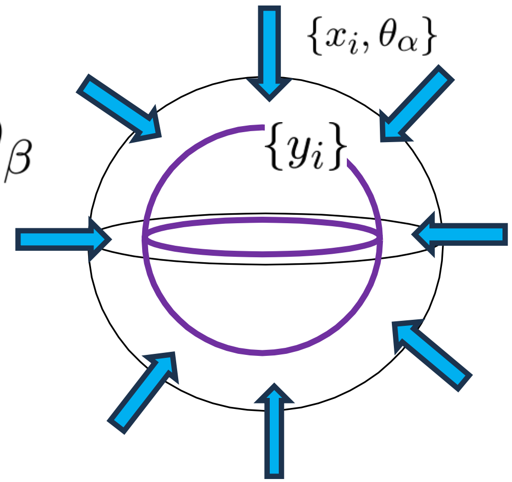
Super Howe duality

Super LLs = Integer LLs + Half-integer LLs



Quantum Mechanics on a supermanifold

$$x_i x_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta = R^2 \quad \Longrightarrow \quad x_i x_i = R^2 - \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta$$



$y_i y_i = R^2$ body-sphere

$$x_i = \sqrt{1 - \frac{1}{R^2}\epsilon_{\alpha\beta}\theta_\alpha\theta_\beta}\; y_i$$

- **Projection to the body**

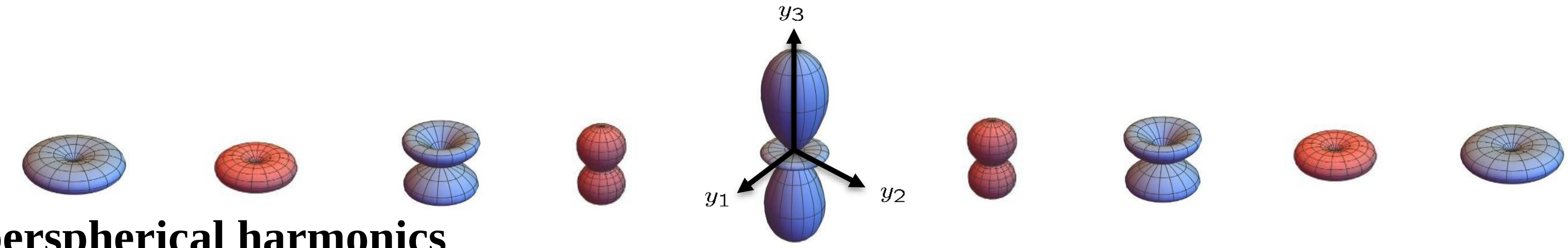
$$\rho(y) = \int d\theta_1 d\theta_2 (1 - \frac{1}{2}\epsilon_{\alpha\beta}\theta_\alpha\theta_\beta)\Psi(x,\theta)^*\Psi(x,\theta)$$

- **Appropriate inner product**
  (redefinition of the metric in SUSY Hilbert space)

$$\int d\theta_1 d\theta_2 (1 - \frac{1}{2}\theta\epsilon\theta) \int_{S^2} d\Omega_2 \Psi^*_{\mathrm{Ghost}}\Psi_{\mathrm{Ghost}} = -1$$

$$\langle \Psi_{\mathrm{Ghost}} | \equiv -\Psi^*_{\mathrm{Ghost}}$$

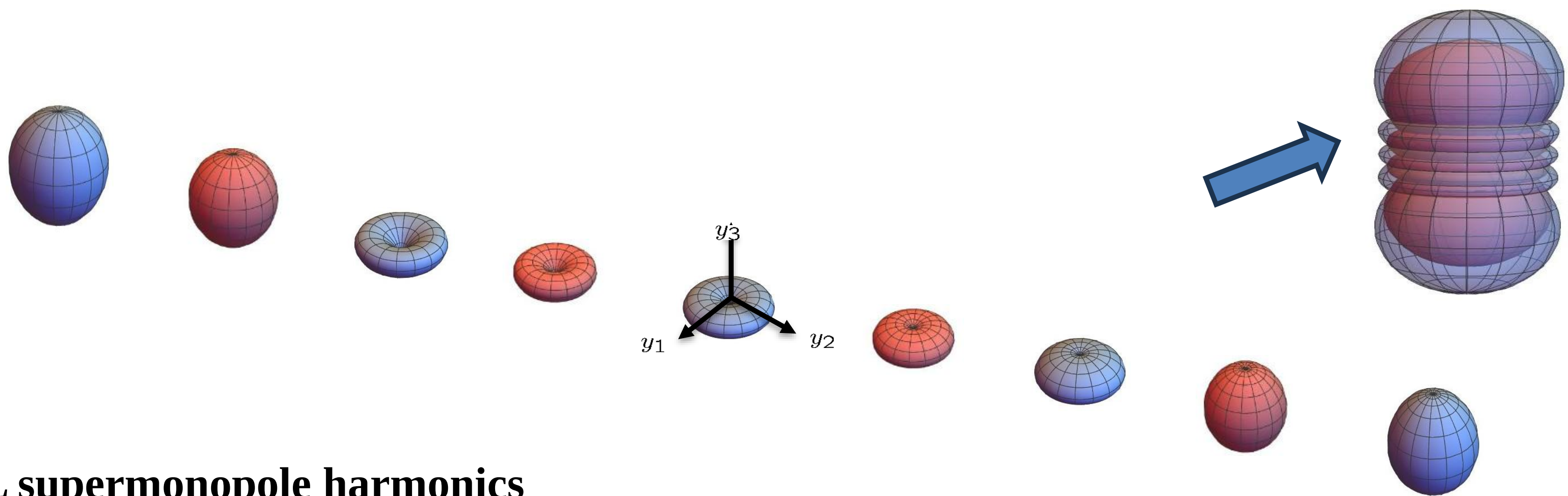
The supermonopole harmonics



Superspherical harmonics

$(l = 2, \ g = 0)$

$m:$ 2 3/2 1 1/2 0 −1/2 −1 −3/2 −2



LLL supermonopole harmonics

$(l = 2, \ g = 2)$

Super quantum matrix geometry

``Bosonic'' monopole harmonics

$$Y^g_{l,m}(x_i, \theta_\alpha)$$

``Fermionic'' monopole harmonics

$$\mathcal{Y}^g_{l,m}(x_i, \theta_\alpha)$$

$$X_i := \begin{pmatrix} \langle Y_l|x_i|Y_l\rangle & \langle Y_i|x_i|\mathcal{Y}_l\rangle \\ \langle \mathcal{Y}_l|x_i|Y_l\rangle & \langle \mathcal{Y}_i|x_i|\mathcal{Y}_l\rangle \end{pmatrix} \quad \Theta_\alpha := \begin{pmatrix} \langle Y_l|x_i|Y_l\rangle & \langle Y_i|x_i|\mathcal{Y}_l\rangle \\ \langle \mathcal{Y}_l|x_i|Y_l\rangle & \langle \mathcal{Y}_i|x_i|\mathcal{Y}_l\rangle \end{pmatrix}$$

$$: \ (4l+1)\times(4l+1) \text{ matrices}$$

Fuzzy supersphere algebra

$$\begin{cases} [X_i, X_j] = i\alpha\epsilon_{ijk}X_k \\ [X_i, \Theta_\alpha] = \alpha\frac{1}{2}(\sigma_i)_{\beta\alpha}\Theta_\beta \\ \{\Theta_\alpha, \Theta_\beta\} = i\alpha\frac{1}{2}(\sigma_2\sigma_i)_{\alpha\beta}X_i. \end{cases}$$

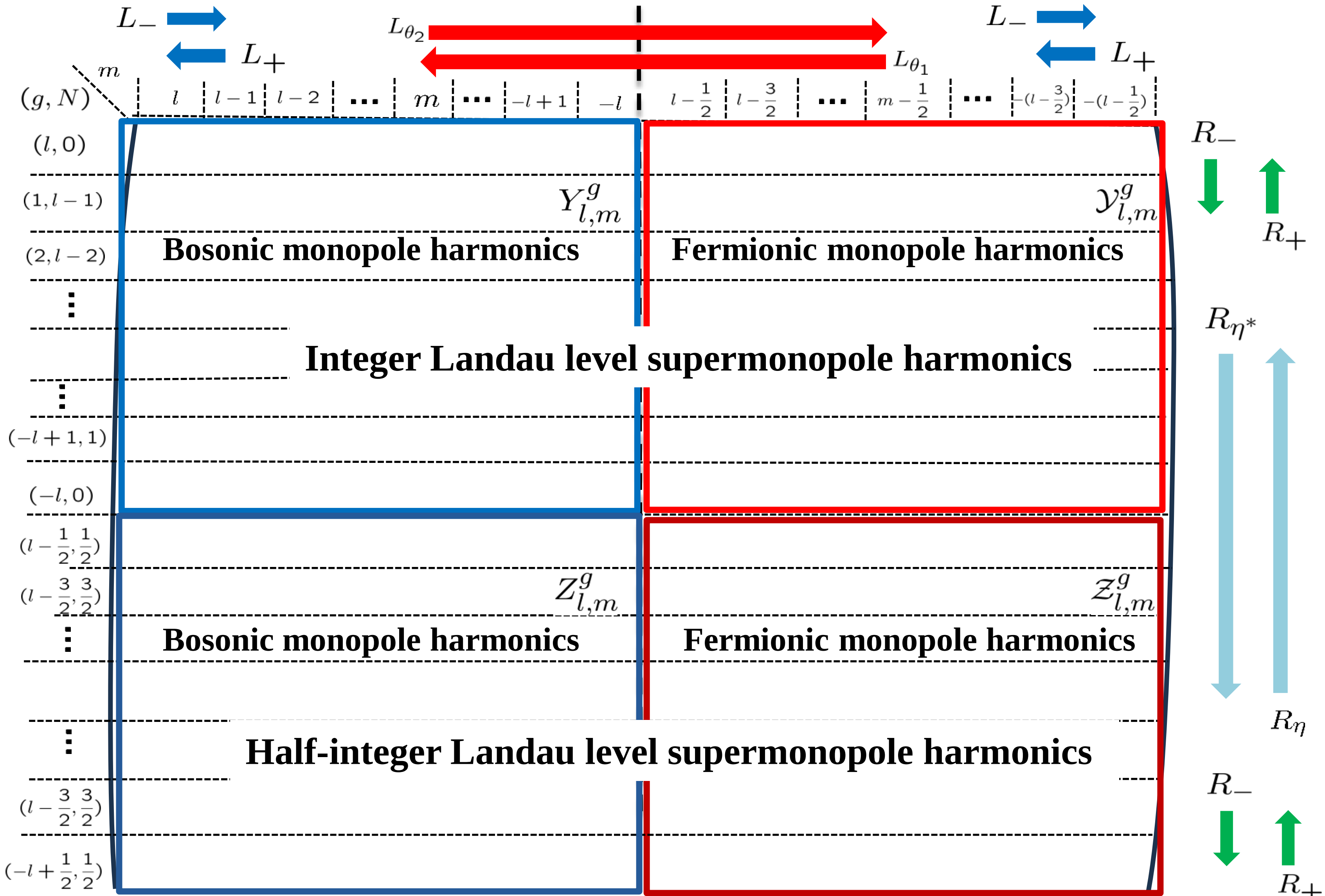
$$\alpha := \frac{g}{l(l+1/2)}$$

$$X_iX_i + \epsilon_{\alpha\beta}\Theta_\alpha\Theta_\beta = \alpha^2 l(l+1/2)1_{4l+1}$$

Super D-matrix

$$\mathcal{D} = e^{-i\chi L_z} e^{-i\theta L_y} e^{-i\phi L_z} e^{2\eta L_{\theta_1} + 2\eta^* L_{\theta_2}}$$

$=$



Super Howe duality

- Dual pair $(UOSp(1|2)_L,\ UOSp(1|2)_R)$

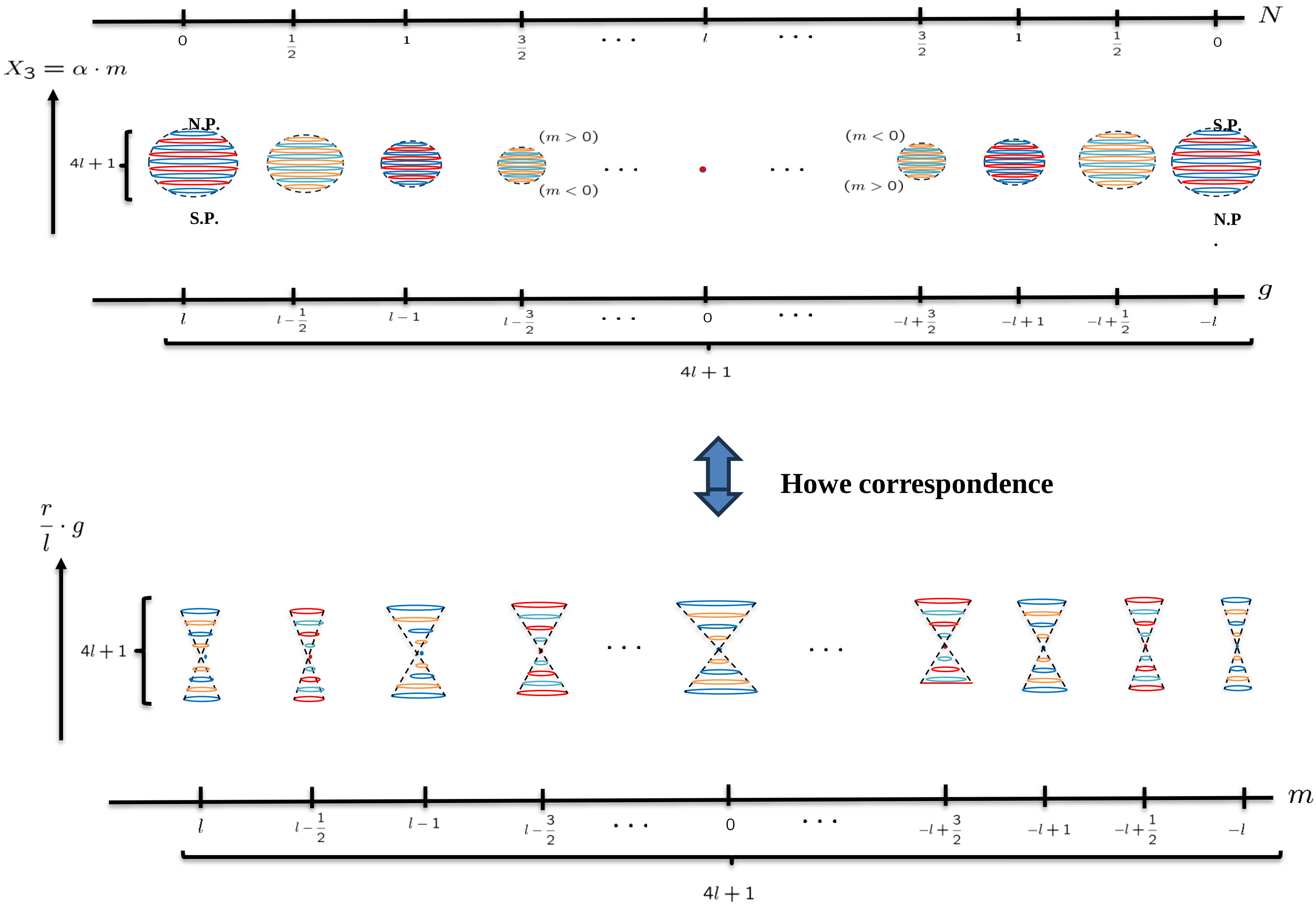
$$\begin{cases} [L_i, L_j] = i\epsilon_{ijk}L_k \\ [L_i, L_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}L_\beta \\ \{L_\alpha, L_\beta\} = i\frac{1}{2}(\sigma_2\sigma_i)_{\alpha\beta}L_i. \end{cases} \qquad \begin{cases} [R_i, R_j] = i\epsilon_{ijk}R_k \\ [R_i, R_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}R_\beta \\ \{R_\alpha, R_\beta\} = i\frac{1}{2}(\sigma_2\sigma_i)_{\alpha\beta}R_i. \end{cases}$$

$$[L_i, R_j] = [L_i, R_\alpha] = [L_\alpha, R_i] = \{L_\alpha, R_\beta\} = 0$$

- Howe-correspondence

$$\mathcal{H} = \oplus_{l=0}^{\infty}(V^{(l)}_{UOSp(1|2)_L} \otimes V^{(l)}_{UOSp(1|2)_R})$$

The Super Howe correspondence



Weinberg の著書 より

S. Weinberg, ``Dream of a Final Theory'' 1994

「究極理論への夢」 小野＋加藤　訳　ダイヤモンド社

第6章　美しい理論

物理理論の美しさは基礎の簡単な原理の上につくられた、剛い数学的構造の中に具現化されるが、***不思議なことに、この種類の美しさをもつ構造は、基礎の原理が誤りと分かった後も生きながらえる傾向にある***。よい例はディラックの電子論である。

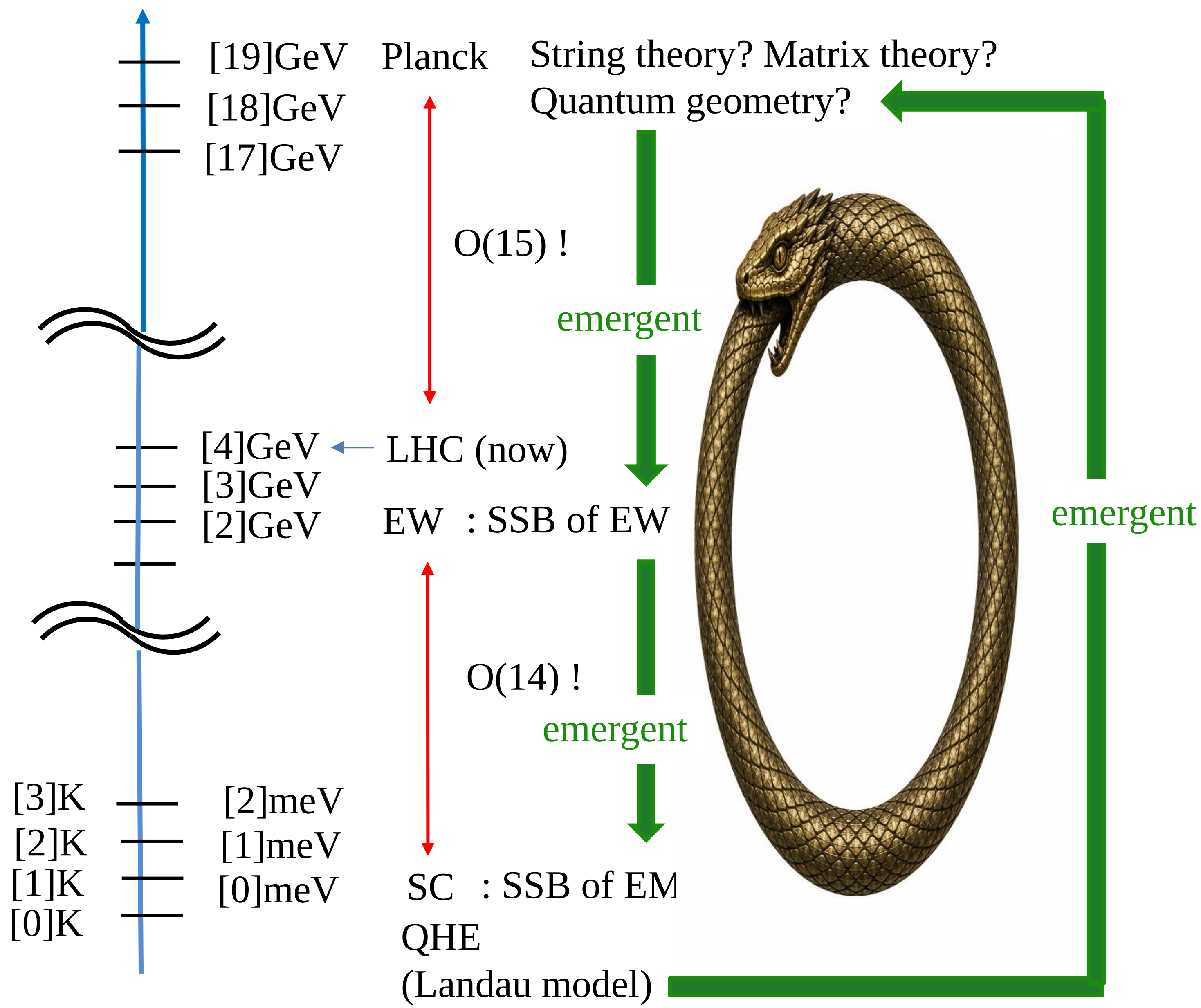
….

その説明になるかもしれないことを、**ボーア**が述べている。

….

彼は、「**自然にあてはめることのできる数学の形式は限られた個数しかない**ので、まったく誤った概念を定式化する過程で、正しい形式を見つけることもあり得る」と言った。

Emergent uroboros from emergent phenomena



Summary

Emergent Quantum Matrix Geometry

Quantum States → Matrix Geometry

(Representation → Geometry)

⇨ *NC ver. ``It from bit''*

Howe duality ***(mathematical structure behind/beyond Landau models)***

Irreps of G ←→ Irreps of G'

Matrix Geometry ←→ Matrix Geometry'

⇨ *Study on the relations among QMGs rather than each QMG*

Relation to Matrix model geometry?

Internal space $\simeq$ External space

momentum A_μ *space-time coordinates*