

# 南部力学の展開：相互作用と統計場の理論

米谷民明  
東京大学

## Developments of Nambu Dynamics — Interactions and statistical field theory—

Tamiaki Yoneya  
Univ. of Tokyo

# Some Backgrounds

H. Awata, M. Li, D. Minic and T. Yoneya, "On the quantization of Nambu brackets", JHEP02(2001)013



T. Yoneya, "Covariantized matrix theory for D-particle", JHEP06(2016)058

T. Yoneya, "Lectures on higher-gauge symmetries from Nambu brackets and covariantized M(atric) theory", in *Noncommutative geometry and physics 4*, (World Scientific, 2017)

T. Yoneya, "Nambu mechanics and Nambu brackets", Butsuri (2017)72, 231 (in Japanese)

T. Yoneya, "Generalized Hamilton-Jacobi theory of Nambu mechanics", PTEP2(2017)023A01

T. Yoneya, "Canonical Nambu mechanics: relevance to string/M theory and approaches to quantization", PTEP12(2021)72, 12C101

T. Yoneya, *Introduction to Nambu's generalized Hamiltonian dynamics*,  
(Springer, April 2026)

T. Yoneya, "Statistical field theory of Interacting Nambu dynamics",  
PTEP2026.053A01



my previous talk in Shimane, 2019

# A review on M(atrrix) theory, its covariantization and related problems

Tamiaki Yoneya

(see also lecture note, arXiv: 1612.08513  
and many previous works referred to, afterward)

- Matrix-theory approach to M theory
- Toward covariantization of *DLCQ* Matrix Theory
- Higher gauge symmetry from Nambu bracket
- 11d covariant action of Matrix theory
- Concluding discussions



What does correspond to  $\delta X^\mu = \{F, G, X^\mu\}_N$ , “Nambu transformation”, in terms of matrices?

We need something with an exact significance even for finite  $N$ !

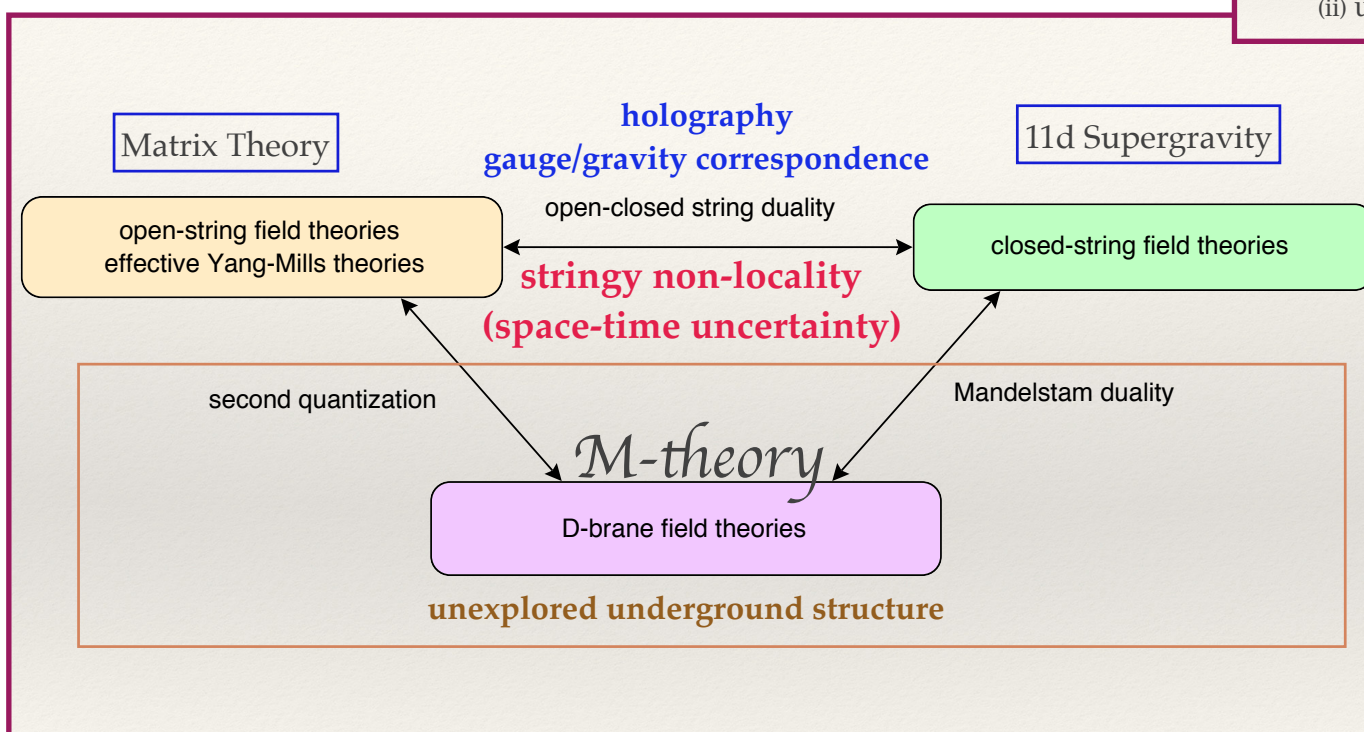
We started this program in the late 90s:  
Unfortunately, however, we could not accomplish it at that time.

Difficulty: *causal time development* should automatically emerge.  
But how?

Our 1999 proposals of **finitely** discretized Nambu brackets :

Awata, Li, Minic and TY, “On the quantization of Nambu bracket”,  
hep-th/9906248; (JHEP 02(2001)013

- (i) using **square** matrices, *augmented by additional variables*
- (ii) using **cubic** matrices with 3 indices  $A_{ijk}$



It seems unfortunate that, in recent years, new development of string/M theory proper seems rather scarce.  
But I hope that people would come back to the real issues of string/M theory, in the not-so distant future.

*Do we have really “fundamental ideas”?*

*Do we have **clues** to experimental verification?*

*Thanks !*



# Motivation for our 2001 work (one of basic motifs from the mid 70's)

Mathematical characterization of the correspondence principle  
between string and GR  
in analogy with that between commutator and Poisson bracket?

$$\alpha' (\propto \ell_s^2 \propto \lambda) \Leftrightarrow \hbar$$

space-time  
uncertainty relation

$$\Delta X \Delta T \gtrsim \ell_s^2$$

T.Y. 1987

For a review, see T.Y. PTP103(2000)1081)

→  $\Delta X \sim g_s^{1/3} \ell_s$  **11D M-theory** scale  $\ell_{11}$

the typical scale of D-particle dynamics in IIA theory in 10 dimensions, M. Li-T.Y. PRL78(1997)1219

$R_{11} = g_s \ell_s$  : compactification radius along the (spatial) 11th direction



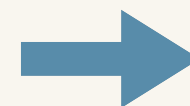
Covariant formulation?

10D Lorentz invariant constraint

$$-\frac{1}{2}([X^\mu, X^\nu])^2 = \lambda^2 I$$

$$\lambda = 4\pi\alpha'$$

$$\frac{1}{2}([X^\mu, X^\nu])^2 \equiv -\sum_{i=1}^9 [X_0, X_i]^2 + \frac{1}{2} \sum_{i,j=1}^9 [X_i, X_j]^2$$



$$\langle ([X^0, X^i])^2 \rangle \geq \lambda^2$$

T.Y. PTP97(1997)949

**Nambu bracket** can be utilized to construct **covariant 11D classical action** for a membrane which is analogous to the Schild action for a 10D string:

$$A_{\text{mem}} = -\frac{1}{\ell_{11}^3} \int d^3\xi \left( \frac{1}{e} \{X^\mu, X^\nu, X^\sigma\}_N \{X_\mu, X_\nu, X_\sigma\}_N - e \right)$$

essentially equivalent to the old **Dirac action** (1962)

$$\{X^\mu, X^\nu, X^\sigma\}_N \equiv \sum_{a,b,c} \epsilon^{abc} \partial_a X^\mu \partial_b X^\nu \partial_c X^\sigma$$



$$\frac{1}{2} \{X^\mu, X^\nu, X^\sigma\} \{X_\mu, X_\nu, X_\sigma\} = -e^2$$

natural extension of the 10D constraint, extending the spacetime uncertainty relation to **11D M-theory**.

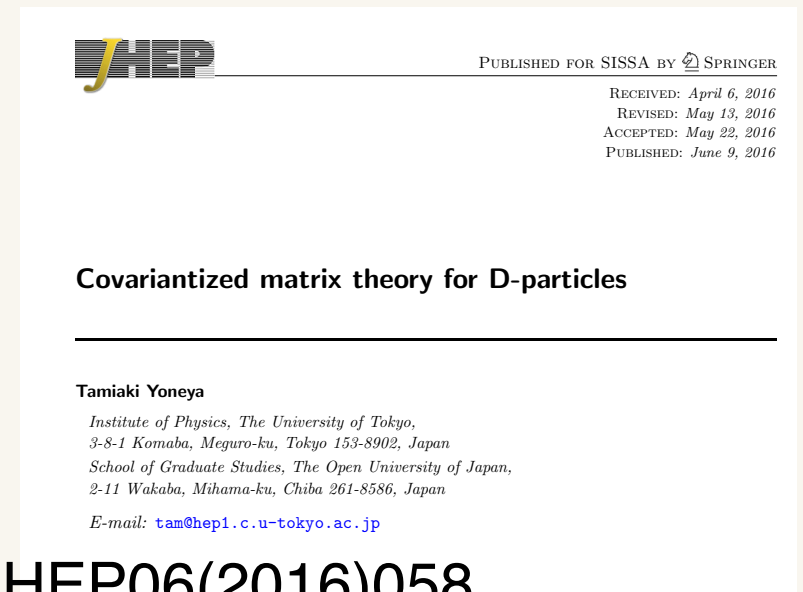
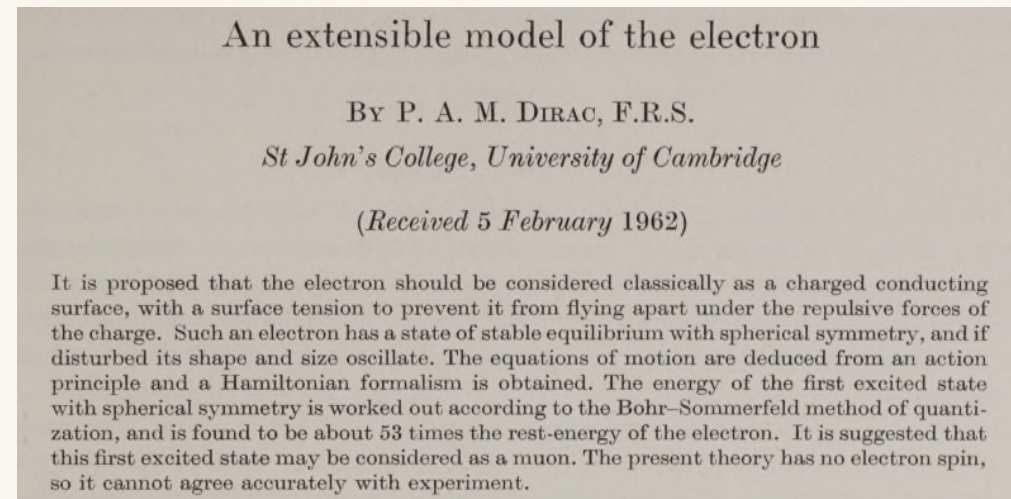


**motivate quantization**

**The volume-preserving diffeo. symmetry for this type of systems can be described by the Nambu equations of motion.**



T. Y., “Covariantized matrix theory for D-particle”, JHEP06(2016)058



All these finally led me to go back to the original motivation of Nambu which was entirely independent of string / M theory.

## Contents of the present talk:

1. What is Nambu dynamics ?
2. Motivations of the present work and some preliminaries
3. Field-theory formalism of many-body Nambu system
4. Statistical ensembles in terms of classical state vectors
5. A solvable model of non-local interaction
6. Equilibrium statistical states
7. Concluding remarks



## **1. What is Nambu dynamics ?**



## Nambu's 1973 paper

## Generalized Hamiltonian Dynamics\*

Yoichiro Nambu

*The Enrico Fermi Institute and the Department of Physics, The University of Chicago, Chicago, Illinois 60637*  
(Received 26 December 1972)

Taking the Liouville theorem as a guiding principle, we propose a possible generalization of classical Hamiltonian dynamics to a three-dimensional phase space. The equation of motion involves two Hamiltonians and three canonical variables. The fact that the Euler equations for a rotator can be cast into this form suggests the potential usefulness of the formalism. In this article we study its general properties and the problem of quantization.

## I. INTRODUCTION

A notable feature of the Hamiltonian description of classical dynamics is the Liouville theorem, which states that the volume of phase space occupied by an ensemble of systems is conserved. The theorem plays, among other things, a fundamental role in statistical mechanics. On the other hand, Hamiltonian dynamics is not the only formalism that makes a statistical mechanics possible. Any set of equations which lead to a Liouville theorem in a suitably defined phase space will do (provided of course that ergodicity may be assumed). With this in mind, let us consider the following scheme.

Let  $(x, y, z) \equiv \vec{r}$  be a triplet of dynamical variables (canonical triplet) which span a three-dimensional phase space. This is a formal generalization of the conventional phase space spanned by a canonical pair  $(p, q)$ . Next introduce two functions,  $H$  and  $G$ , of  $(x, y, z)$ , which serve as a pair of "Hamiltonians" to determine the motion of points in phase space. More precisely, we postulate the following "Hamilton equations":

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial(H, G)}{\partial(y, z)}, \\ \frac{dy}{dt} &= \frac{\partial(H, G)}{\partial(z, x)}, \\ \frac{dz}{dt} &= \frac{\partial(H, G)}{\partial(x, y)}, \end{aligned} \quad (1)$$

or in a vector notation

$$\frac{d\vec{r}}{dt} = \vec{\nabla}H \times \vec{\nabla}G. \quad (1')$$

$[F, H, G]$ . Obviously a PB is antisymmetric under interchange of any pair of its components. As a result we have  $H = F = 0$ , i.e., both  $H$  and  $G$  are constants of motion. The orbit of a system in phase space is thus determined as the intersection of two surfaces,  $H = \text{const.}$  and  $G = \text{const.}$

Equation (1) or (1') also shows that the velocity field  $d\vec{r}/dt$  is divergenceless,

$$\vec{\nabla} \cdot (\vec{\nabla}H \times \vec{\nabla}G) = 0, \quad (3)$$

and this amounts to a Liouville theorem in our phase space.

The above properties immediately tempt us to construct a statistical mechanics where a canonical ensemble is characterized by a generalized Boltzmann distribution in phase space with a weight factor

$$e^{-\beta H - \gamma G}.$$

Two temperaturelike intensive parameters are thus required to specify the ensemble, much as in a grand canonical ensemble.

It is obvious that this kind of generalization can be extended to a phase space of any dimensionality,  $n$ . We would introduce an  $n$ -component vector  $x_i$  and  $n-1$  Hamiltonians  $H_k$ , and postulate in lieu of Eqs. (1) and (2)

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{j,k \dots l} \epsilon_{ijk \dots l} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} \dots \frac{\partial H_{n-1}}{\partial x_l}, \\ \frac{dF}{dt} &= \frac{\partial(F, H_1, H_2, \dots, H_{n-1})}{\partial(x_1, x_2, \dots, x_n)}, \end{aligned} \quad (4)$$

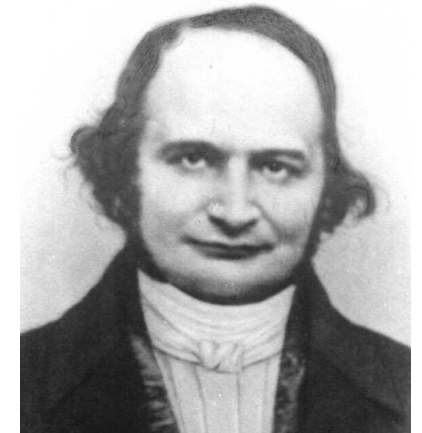
where  $\epsilon_{ijk \dots l}$  is the Levi-Civita tensor.

From the standpoint of physics, however, we



Y. Nambu  
at Marshak conf,  
CCNY, 1977

**Historically, Nambu-type structure has been conceived of by Jacobi in the 1840s in his lectures.**



# VORLESUNGEN ÜBER DYNAMIK

VON

C. G. J. JACOBI

NEBST

FÜNF HINTERLASSENEN ABHANDLUNGEN DESSELBEN

HERAUSGEGEBEN

VON

A. CLEBSCH.

UNTER BEFÖRDERUNG DER KÖNIGLICH PREUSSISCHEN AKADEMIE  
DER WISSENSCHAFTEN.

B E R L I N.

DRUCK UND VERLAG VON GEORG REIMER.

1866.

Dehnen wir die Theorie dieses Multiplikators  $M$  auf ein System zweier simultanen Differentialgleichungen zwischen drei Variablen aus. Dasselbe sei in folgender Form vorgelegt:

$$(2.) \quad dx : dy : dz = X : Y : Z,$$

die Integralgleichungen nach den willkürlichen Constanten aufgelöst seien

$$(3.) \quad f = \alpha, \quad \varphi = \beta;$$

dann hat man

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0, \quad \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = 0,$$

und hieraus ergibt sich

$$dx : dy : dz = \frac{\frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial y}}{\frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial z}} : \frac{\frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial z}}{\frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial x}}.$$

Setzt man

$$A = \frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial y}, \quad B = \frac{\partial f}{\partial z} \frac{\partial \varphi}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial z}, \quad C = \frac{\partial f}{\partial x} \frac{\partial \varphi}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial \varphi}{\partial x},$$

so ist also

$$dx : dy : dz = A : B : C,$$

|   |                                                                                              |     |
|---|----------------------------------------------------------------------------------------------|-----|
| 1 | Introduction: Historical Overview and Basic Concepts .....                                   | 1   |
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# Introduction to Nambu’s Generalized Hamiltonian Dynamics

published in April, 2026

In this small book, however, the statistical mechanics of Nambu dynamics, the subject of today’s talk, is not touched upon.



# Nambu's proposal:

Hamilton equations of motion

$$\frac{dx^i}{dt} = X^i(x), \quad (i = 1, \dots, 2n)$$

$$\{x^i; (x^1, \dots, x^{2n}) \equiv (p_1, q_1, p_2, q_2, \dots, p_n, q_n)\}$$

even-dimensional phase space

$$X^{2i-1}(x) = -\frac{\partial H}{\partial q_i}, \quad X^{2i}(x) = \frac{\partial H}{\partial p_i}$$

a single Hamiltonian  $H$

Nambu equations of motion

$$\frac{dx^i(t)}{dt} = X^i(x(t))$$

$$(x^1(t), x^2(t), x^3(t))$$

3-dimensional phase space

$$X^i(x) = \epsilon^{ijk} \frac{\partial K}{\partial x^j} \frac{\partial H}{\partial x^k}$$

two Hamiltonians on an equal footing

$$H = H(x^1, x^2, x^3), \quad K = K(x^1, x^2, x^3)$$

$$\dot{x}^i = \{K, H, x^i\} = \frac{1}{2} \epsilon^{ijk} \frac{\partial(K, H)}{\partial(x^j, x^k)}$$

canonical Nambu bracket relation:

$$\dot{F} = \partial_i F \dot{x}^i = \{K, H, x^i\} \partial_i F = \{K, H, F\}$$

$$\{x^i, x^j, x^k\} = \epsilon^{ijk}$$

## Example: Euler equation of a rigid rotator

phase space:  $(x^1, x^2, x^3) = (L_1, L_2, L_3)$

$$\frac{dL_1}{dt} = -\left(\frac{1}{I_2} - \frac{1}{I_3}\right) L_2 L_3$$

$$\frac{dL_2}{dt} = -\left(\frac{1}{I_3} - \frac{1}{I_1}\right) L_3 L_1$$

$$\frac{dL_3}{dt} = -\left(\frac{1}{I_1} - \frac{1}{I_2}\right) L_1 L_2$$

$$K = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2)$$

$$H = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right)$$

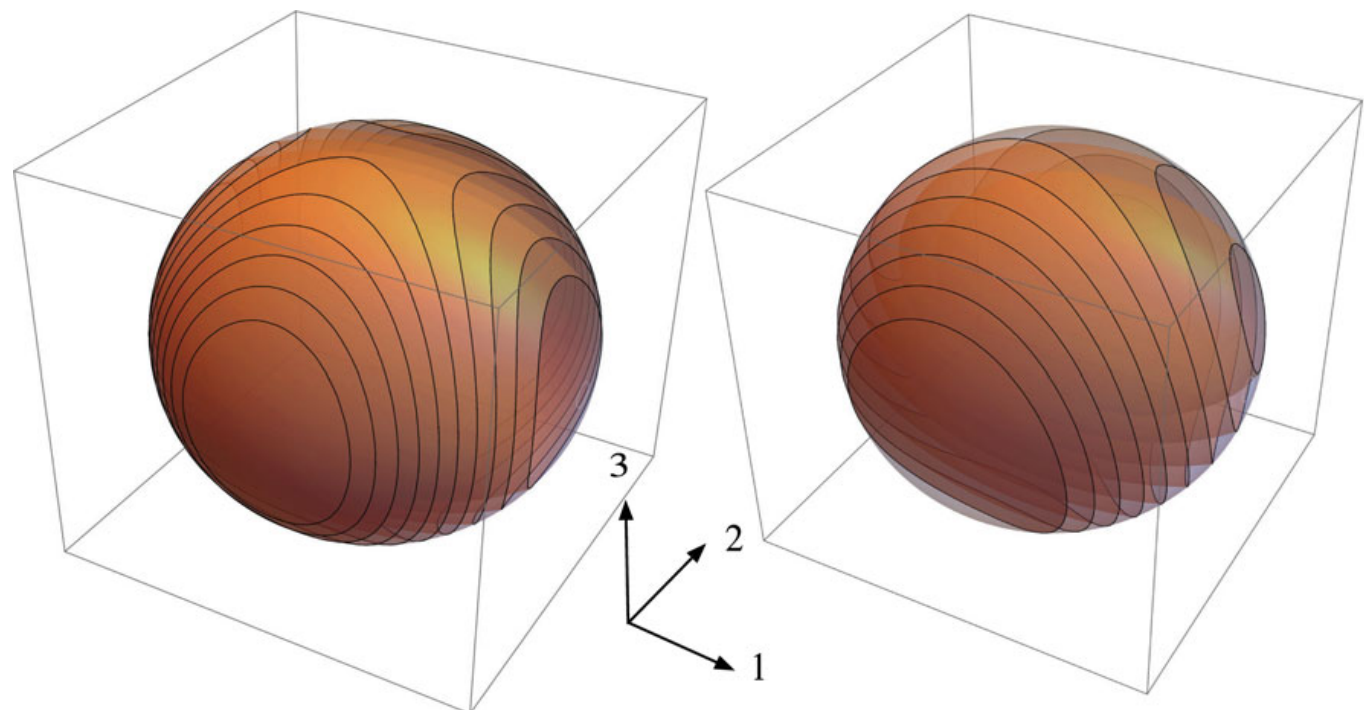
slight generalization

$$H = \frac{1}{2}(a(x^1 - e)^2 + b(x^2 - f)^2 + c(x^3 - g)^2)$$

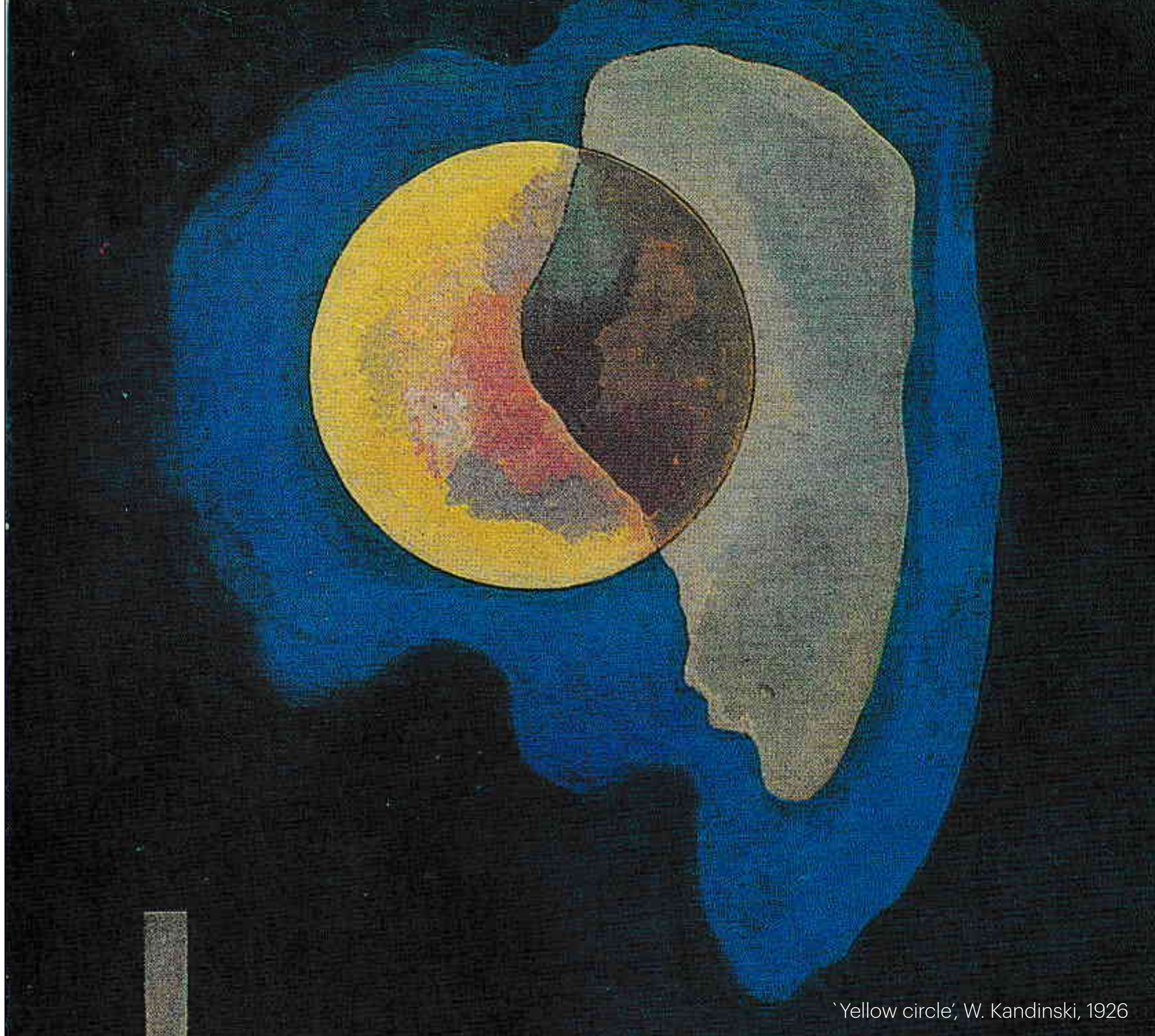
**Fig. 2.4** The trajectories of the solutions of the equations of motion

$$e = f = g = 0$$

$$e = f = g > 0$$







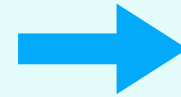
'Yellow circle', W. Kandinski, 1926



# Variational principle: characterizing the difference of Nambu dynamics from the usual Hamiltonian dynamics

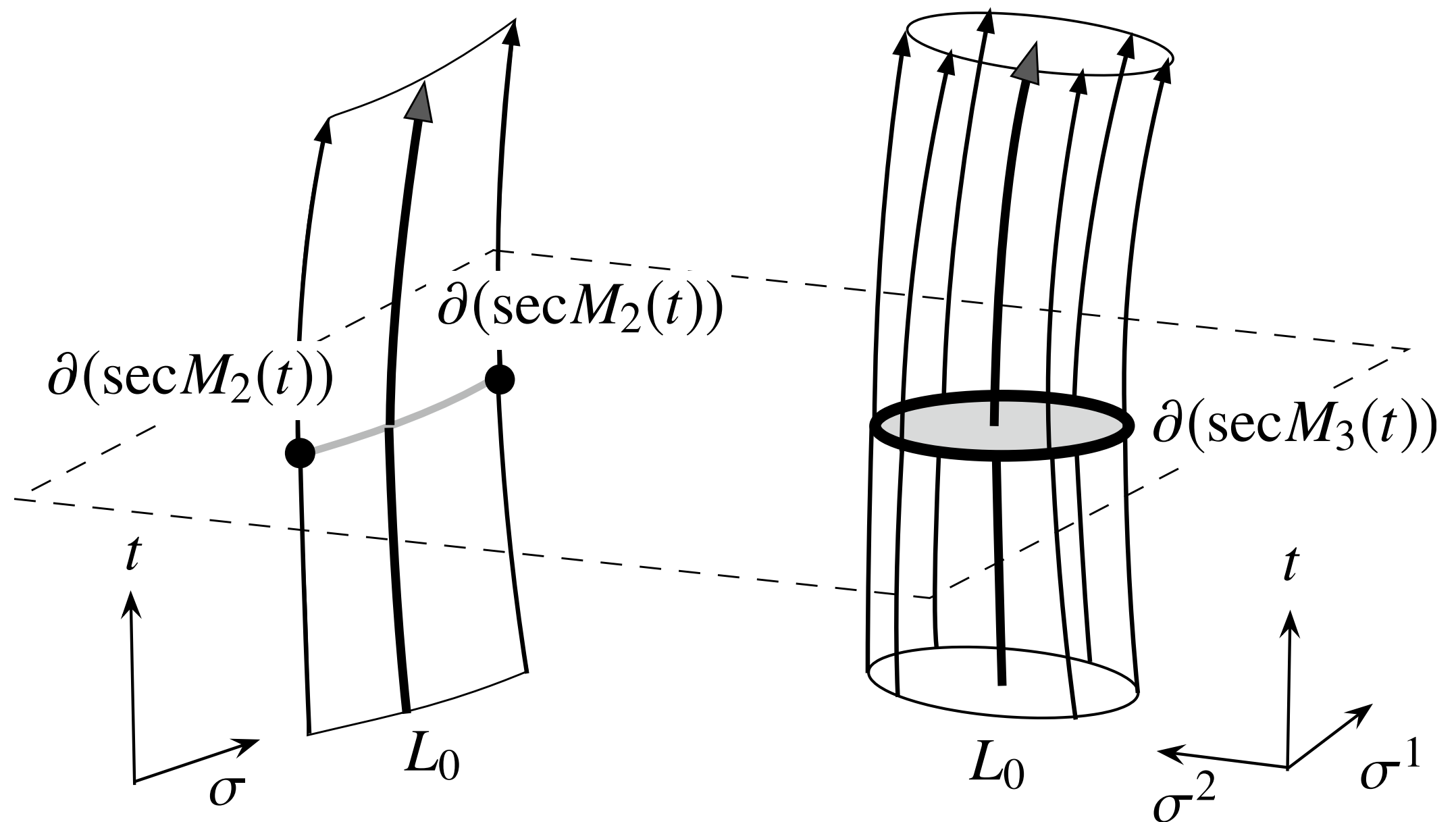
Hamiltonian dynamics

$$dp_i(t; \sigma) \wedge dq_i(t; \sigma) - dH(p, q) \wedge dt$$



Nambu dynamics

$$dx^1 \wedge dx^2 \wedge dx^3 - dK \wedge dH \wedge dt$$



**Fig. 3.1** The infinitesimal variations corresponding to  $M_2$  (left) and  $M_3$  (right)

In general  $n$  dimensions  $(x^1, x^2, \dots, x^n)$  ,  $n-1$  Hamiltonians

$$(H_1, H_2, \dots, H_{n-1})$$

$$\omega^{(n)} = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n - dH_1 \wedge dH_2 \wedge \dots \wedge dH_{n-1} \wedge dt$$

$$\begin{aligned} \dot{x}^i &= \epsilon^{ij_1 \dots j_{n-1}} \partial_{j_1} H_1 \partial_{j_2} H_2 \dots \partial_{j_{n-1}} H_{n-1} = \frac{\partial(H_1, H_2, \dots, H_{n-1}, x^i)}{\partial(x^1, x^2, \dots, x^n)} \\ &\equiv \{H_1, H_2, \dots, H_{n-1}, x^i\}. \end{aligned}$$

Poisson bracket is generalized to Nambu bracket

$$\{F_1, F_2, \dots, F_n\} \equiv \epsilon^{i_1 i_2 \dots i_n} \frac{\partial F_1}{\partial x^{i_1}} \frac{\partial F_2}{\partial x^{i_2}} \dots \frac{\partial F_n}{\partial x^{i_n}} = \frac{\partial(F_1, F_2, \dots, F_n)}{\partial(x^1, x^2, \dots, x^n)}$$

The trajectories of Nambu particle = the loci of intersection of the level surfaces of given Hamiltonians



## **2. Motivation of the present work**



## Nambu's original motivation

- **Extended statistical mechanics with multiple temperatures**

“Hamiltonian dynamics is not the only formalism that makes a statistical mechanics possible. Any set of equations which lead to a Liouville theorem in a suitably defined phase space will do.....

The above properties immediately tempt us to construct a statistical mechanics where a canonical ensemble is characterized by a generalized Boltzmann factor in phase space with a weight factor

$$e^{-\beta H - \gamma G}.$$

Two temperature-like intensive parameters are thus required to specify the ensemble, .....

**However, there is a tacit but crucial assumption here that the systems can interact with each other, however weak it may be, since the exchange of energies among systems in the ensemble are required to exist, in order to reach an equilibrium states.**

Unfortunately, Nambu failed to achieve his goal:

“We are thus led to a rather unexciting result that the only continuous canonical transformations consist of independent transformations of the individual triplets. This contrasts with the case of usual canonical doublets where a general transformation involve all the variables simultaneously.”

In other words,



**It is impossible to extend Nambu dynamics to a higher-dimensional system, at least, if we require that, as in the usual Hamiltonian formalism, canonical transformations are extended for mixing different canonical Nambu coordinates.**

$$(x^1, x^2, \dots, x^n) \in \mathbb{R}^n \quad \longrightarrow \quad (x_a^1, x_a^2, \dots, x_a^n) \quad (a = 1, 2, \dots, N) \in \mathbb{R}^{Nn}$$

$$\{x_a^{i_1}, x_a^{i_2}, \dots, x_a^{i_n}\} = \epsilon^{i_1 i_2 \dots i_n}$$

Nambu bracket is much more rigid with respect to its canonical structure than Poisson bracket: phase space with different  $a$  cannot be mixed.

This poses a **fundamental difficulty in introducing interactions within the framework of Nambu dynamics**:

Interactions would require necessarily that the equations of motion describe **canonical transformations which mix different Nambu coordinates along with dynamical evolutions**.

Clearly, if there were no interactions whatsoever, it would be never possible to discuss approaches to equilibrium state,

and consequently nor the basic concepts of thermodynamics such as temperatures, and heat bath, etc.

N.B.

**In start contrast to this, there is no such difficulty in ordinary Hamiltonian dynamics, since we can introduce interactions in the Hamiltonian itself.**



However, canonical transformation preserving each Nambu multiplet can be formulated consistently: in infinitesimal form it is

$$x^i \rightarrow x'^i = x^i + \varepsilon X^i(x), \quad \partial_i X^i = 0 \quad \mathcal{D}_X \equiv X^i \partial_i$$

the consistency of this structure is guaranteed by the generalized Jacobi identity:

$$\begin{aligned} \mathcal{D}_X \{F_1, \dots, F_n\} &= \{\mathcal{D}_X F_1, F_2, \dots, F_n\} + \{F_1, \mathcal{D}_X F_2, \dots, F_n\} \\ &+ \dots + \{F_1, \dots, F_{n-1}, \mathcal{D}_X F_n\} \end{aligned}$$

In the present work, we shall develop a **new framework of interacting Nambu dynamics** by employing the method of quantum-field theory, and establish a possible **classical statistical mechanics of Nambu dynamics**.

Our purpose is

*to justify Nambu's expectations from an entirely dynamical standpoint.*

An Aside

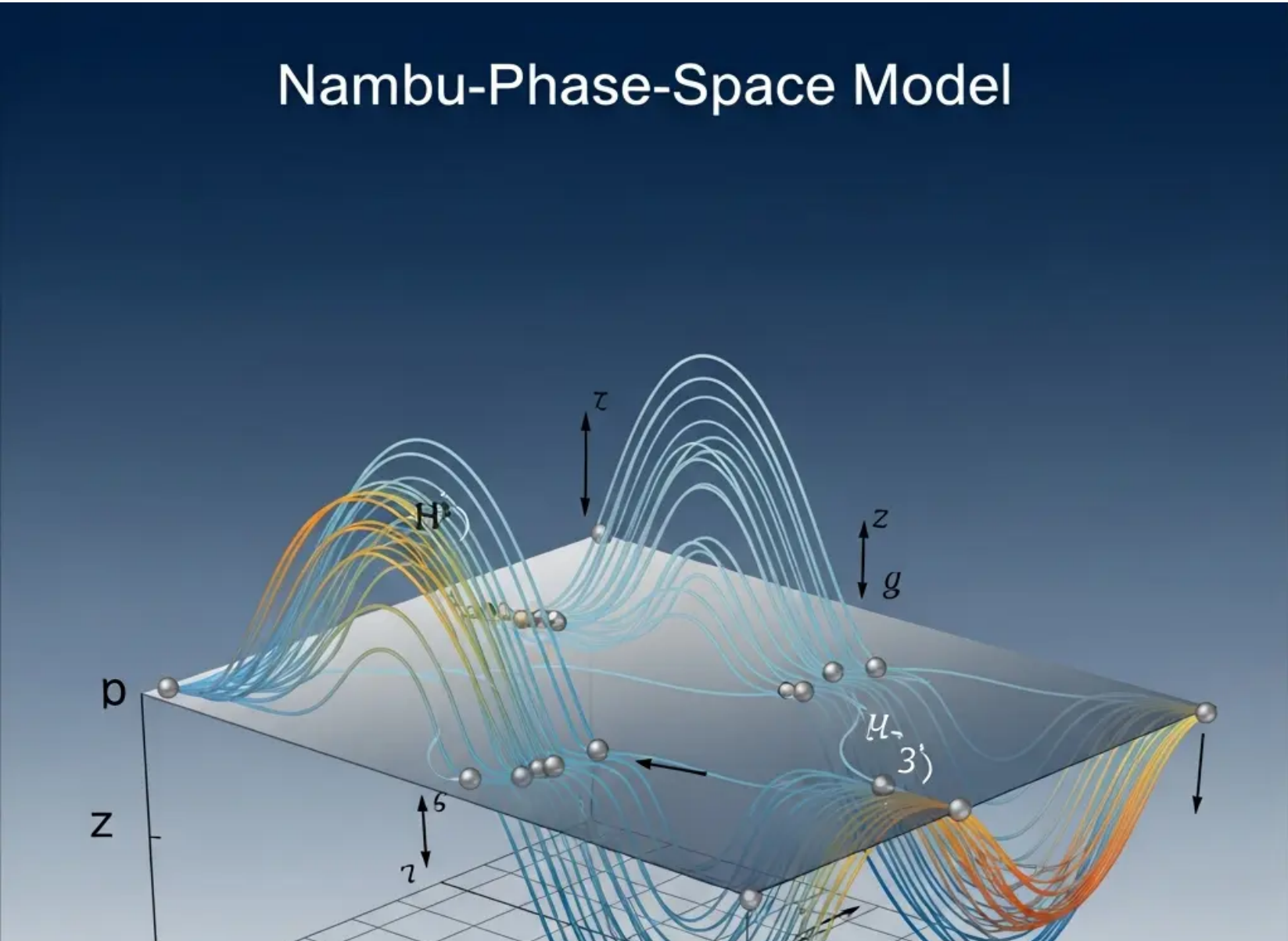
A web page *by a third party*, introducing the present work, only a few days after submitting it to the arXiv.

PHYSICS

# Nambu Dynamics: Statistical Field Theory Advances



DECEMBER 13, 2025  
BY MUHAMMAD ROHAIL T.



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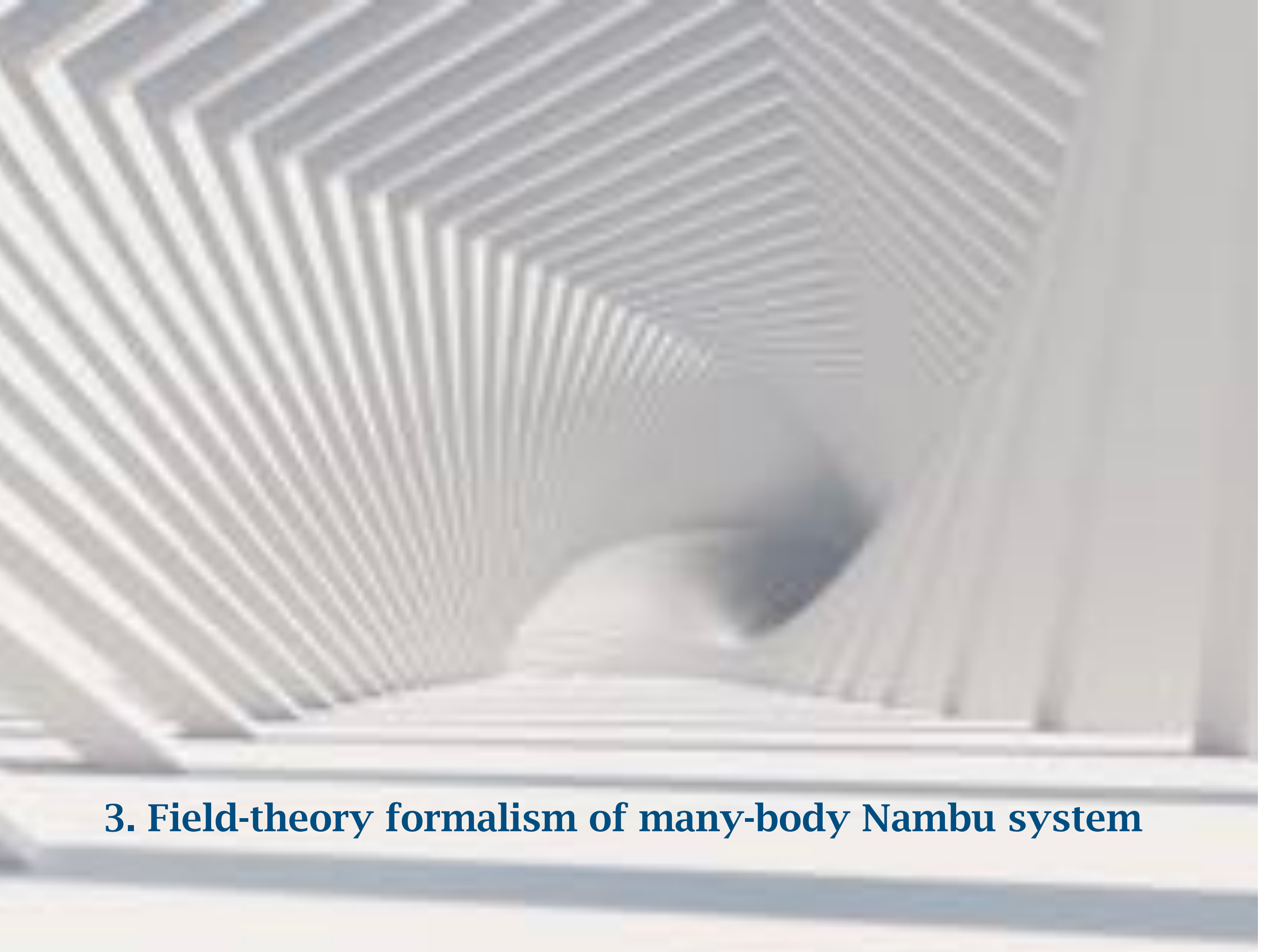
Quantum Effects Boost Room-Temperature Thermoelectricity in Weyl Semimetals

Formal Semantics Developed for Core Continuous-Variable Quantum Language



“Classical dynamics faces a long-standing challenge in extending the Nambu framework to incorporate interactions without disrupting its fundamental mathematical structure, and Tamiaki Yoneya from the University of Tokyo .... now addresses this problem with a novel statistical field theory..... ”

“The core of this achievement lies in formulating a field theory on Nambu’s generalized field space, utilizing an operator formalism to describe the system’s evolution....”



### **3. Field-theory formalism of many-body Nambu system**

# Statistical Field Theory of Interacting Nambu Dynamics

Tamiaki Yoneya

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Received December 19, 2025; Revised March 16, 2026; Accepted March 27, 2026; Published April 3, 2026

.....  
We develop a statistical field theory for classical Nambu dynamics by partially employing the quantum field-theory method. One of the unsolved problems in Nambu dynamics has been to extend it to interacting systems without violating a generalized canonical structure characterized by the presence of multiple Hamiltonians, which together govern the dynamics of time evolution on an equal footing through a Nambu bracket as an analog to the Poisson bracket of ordinary Hamiltonian dynamics with a single Hamiltonian. In the present paper, we propose to include interactions from the standpoint of classical statistical dynamics by formulating it as a field theory on Nambu’s generalized phase space in an operator formalism. We first construct a general framework for such a field theory and its probabilistic interpretation. Then, on the basis of this framework, we give an exactly solvable model of self-interactions in a many-body Nambu system treated as a closed dynamical system satisfying the H-theorem. It is shown that a generalized microcanonical ensemble and a generalized canonical ensemble characterized by many temperatures are reached dynamically as equilibrium states starting with certain classes of initial nonequilibrium states via a continuous Markov process, which lies beyond the usual framework based on the Liouville theorem. Compared with the usual classical statistical mechanics on the basis of standard Hamiltonian dynamics, some important new features associated with Nambu dynamics will emerge, with respect to the symmetries underlying the dynamics of the nonequilibrium as well as the equilibrium states and also to some conceptual properties, such as a formulation of a generalized Kubo–Martin–Schwinger-like condition characterizing the generalized canonical equilibrium states and the “relative” nature of the temperatures.  
.....

Subject Index A00, A02, B37, B86, B87

Two preliminaries toward a field-theory formulation:

*Nambu equations of motion in terms of the Liouville equation*

$$\frac{\partial \rho}{\partial t} + X^i(x) \partial_i \rho = 0, \quad X^i(x) \equiv \epsilon^{ijk} \partial_j H(x) \partial_k K(x)$$

→  $\left( \frac{\partial}{\partial t} + X^i(x) \frac{\partial}{\partial x^i} \right) G_r(x, t; x_0, t_0) = \delta^3(x - x_0) \delta(t - t_0)$

Green function

$$G_r(x, t; x_0, t_0) = \delta^3(x - \underline{x(t - t_0; x_0)}) \quad \text{unique solution of the Nambu equations of motion}$$

The Liouville equation, multiplied by the imaginary unit can be regarded as Schrödinger equation with Hamiltonian  $H_0 \equiv \pi_i X^i(x)$   $\pi_i \equiv -i\hbar \partial_i$

**Classical theory, but can be treated formally and partially as a quantum system !**

*The  $\mathcal{N}$ -symmetry*

$$X^i(x) \equiv \epsilon^{ijk} \partial_j H(x) \partial_k K(x)$$

invariant under the transformation  $(H, K) \rightarrow (H', K')$

$$dH \wedge dK - dH' \wedge dK' = d(HdK - H'dK') = 0$$



### 3.1. *Field operators for Nambu particles*

$\psi(x, t)$ ,  $\psi^\dagger(x, t)$  as *basic dynamical degrees of freedom*  
defined on the “ $\mu$ -space” of 3D Nambu phase space

$$[\psi(x, t), \psi^\dagger(y, t)] = \delta^3(x - y), \quad [\psi(x, t), \psi(y, t)] = 0 = [\psi^\dagger(x, t), \psi^\dagger(y, t)],$$

$$\psi(x, t)|0\rangle = 0 = \langle 0|\psi^\dagger(x, t).$$

field equations  $(\partial_t + X^i(x)\partial_i)\psi(x, t) = 0 = (\partial_t + X^i(x)\partial_i)\psi^\dagger(x, t)$



$$G_r(x, t : x_0, t_0) = \langle 0|T(\psi(x, t)\psi^\dagger(x_0, t_0))|0\rangle = [\psi(x, t), \psi^\dagger(x_0, t_0)]\theta(t - t_0)$$

$$\psi(x, t) = e^{\hat{\mathcal{H}}_0(t-t_0)}\psi(x, t_0)e^{-\hat{\mathcal{H}}_0(t-t_0)}, \quad \psi^\dagger(x, t) = e^{\hat{\mathcal{H}}_0(t-t_0)}\psi^\dagger(x, t_0)e^{-\hat{\mathcal{H}}_0(t-t_0)}$$

$$\hat{\mathcal{H}}_0 \equiv \int \psi^\dagger(x, t)X^i(x)\partial_i\psi(x, t)d^3x = - \int \partial_i\psi^\dagger(x, t)X^i(x)\psi(x, t)d^3x = -\mathcal{H}_0^\dagger.$$

(N.B.: No imaginary unit)

**Nambu dynamics can equivalently be formulated as a quantum field theory !**

### 3.2. Symmetry transformations and the conservation laws in terms of field operators

$$[\hat{\mathcal{H}}_0, \hat{H}] = 0 = [\hat{\mathcal{H}}_0, \hat{K}]$$

$$\hat{H} = \int \psi^\dagger(x, t) H(x) \psi(x, t) d^3x,$$

$$\hat{K} = \int \psi^\dagger(x, t) K(x) \psi(x, t) d^3x$$

$$\begin{aligned} \psi(x, t) &\rightarrow e^{-i\lambda(H(x), K(x))} \psi(x, t) \\ \psi^\dagger(x, t) &\rightarrow e^{i\lambda(H(x), K(x))} \psi^\dagger(x, t) \end{aligned} \quad \lambda(x, y) = c_1 x + c_2 y + c_0$$

linearized  $\mathcal{N}$ -symmetry  $\text{SL}(2, \mathbb{R})$

$$\begin{pmatrix} H' \\ K' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H \\ K \end{pmatrix}, \quad ad - bc = 1$$

the same transformation laws for the operators  $(\hat{H}, \hat{K})$ .

Keeping these symmetries:

a guiding principle for constructing interacting Nambu dynamics

### 3.3. Basis states for classical many-body Nambu systems

$$\begin{aligned}
 |[x], t\rangle_N &\equiv \frac{1}{\sqrt{N!}} \Psi_N^\dagger([x], t) |0\rangle, & \Psi_N^\dagger([x], t) &\equiv \prod_{a=1}^N \psi^\dagger(x_a, t). \\
 {}_N\langle [x], t| &\equiv \frac{1}{\sqrt{N!}} \langle 0| \Psi_N([x], t), & \Psi_N([x], t) &\equiv \prod_{a=1}^N \psi(x_a, t), \\
 1 &= \sum_{N=0}^{\infty} \int |[x], t\rangle_N {}_N\langle [x], t| [dx]_N, & [dx]_N &\equiv \prod_{a=1}^N d^3x_a \\
 {}_N\langle [x], t| [x'], t\rangle_{N'} &= \delta_{NN'} \frac{1}{N!} \sum_{P(a)} \prod_{a=1}^N \delta^3(x_a - x'_{P(a)})
 \end{aligned}$$

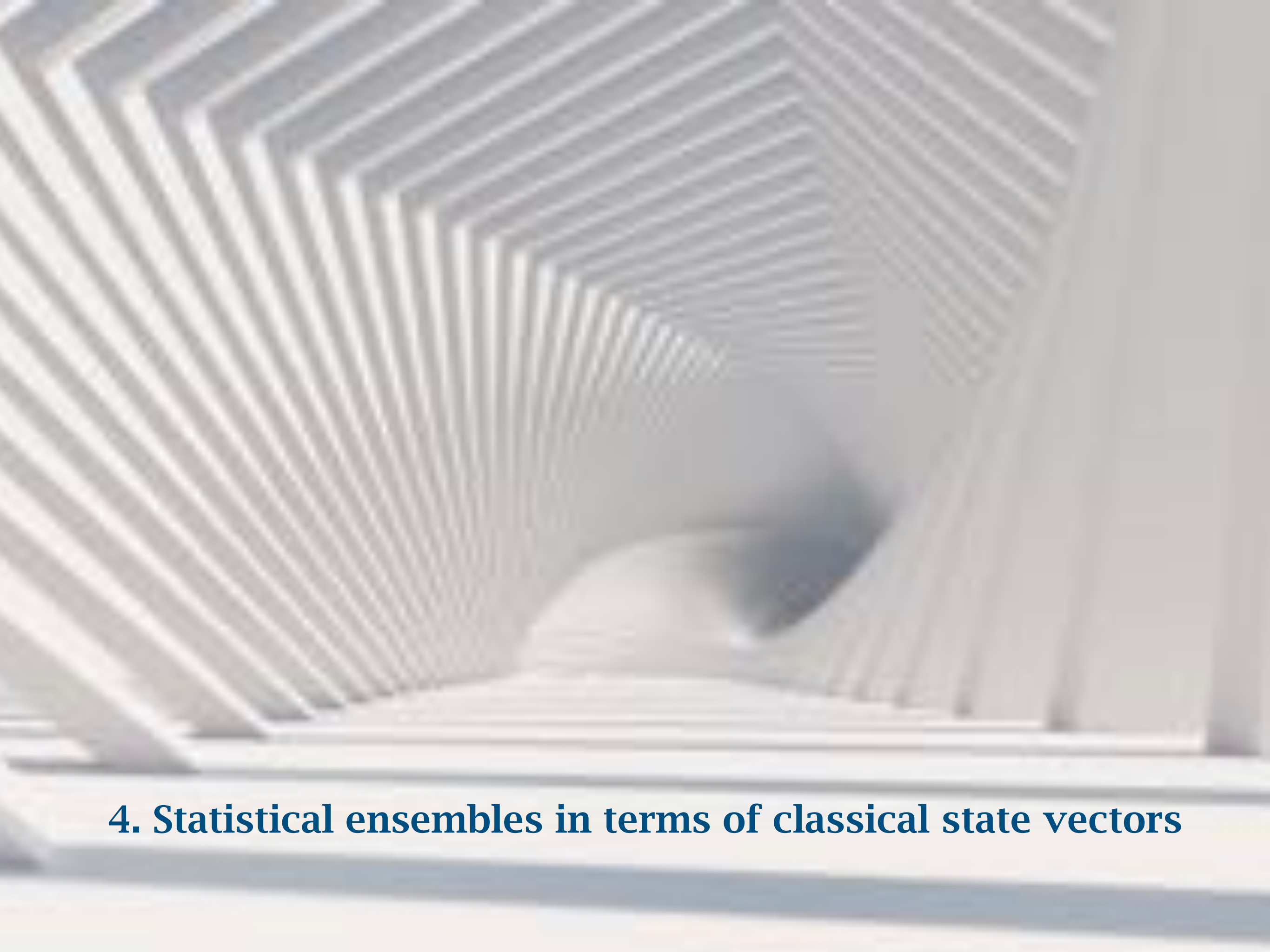
a characterization of the basis states: eigenstates of the density operator

$$\hat{\rho}(x, t) = \psi^\dagger(x, t) \psi(x, t)$$

$$\hat{\rho}(x, t) |[x], t_0\rangle_N = \rho(x, t) |[x], t_0\rangle_N, \quad \rho(x, t) = \sum_{a=1}^N \delta^3(x - x_a(t - t_0; x_a))$$

precise and maximum information on many-body states of Nambu particles  
even classically a highly idealized concept !!

It would be more natural and general to treat states  
with certain *statistical* spreading for each particle by introducing statistical ensembles



## **4. Statistical ensembles in terms of classical state vectors**



#### 4.1. Classical probability interpretation

Given an arbitrary, real nonnegative function  $f[x] = f(x_1, x_2, \dots, x_N) (\geq 0)$   
*totally symmetric* under arbitrary permutations of  $N$  coordinates

classical statistical state  $|f(t_0)\rangle \equiv \int f[x] |[x], t_0\rangle [dx]$

$$\langle [x], t_0 | f(t_0) \rangle = \int f[x'] \langle [x], t_0 | [x'], t_0 \rangle [dx'] = f[x]$$

normalization condition  $\int f[x] [dx] = \iint f[x'] \langle [x], t_0 | [x'], t_0 \rangle [dx'] [dx] = 1$



$f[x]$  probability distribution of  $N$  Nambu particles

with respect to the  $3N$ -dimensional “complexion”  $[x] = (x_1, x_2, \dots, x_N)$

- No interference effect
- Fermionic field is excluded

(both by construction)

## 4.2. *Further properties of the statistical states*

statistical state in general has a nontrivial time dependence,  
because of the microscopic streaming of Nambu particles

$$\begin{aligned}\partial_{t_0}|f(t_0)\rangle &= \hat{\mathcal{H}}_0|f(t_0)\rangle \\ &= \int \sum_{a=1}^N \left( X^i(x_a) \frac{\partial f[x]}{\partial x_a^i} \right) |[x], t_0\rangle [dx]\end{aligned}$$

More explicitly, the evolution of a generic statistical state is given as

$$\begin{aligned}\langle [y], t | f(t_0) \rangle &= \int \sum_{P(a)} f[x] \frac{1}{N!} \left[ \prod_{a=1}^N \delta^3(y_{P(a)} - x(t - t_0; x_a)) \right] [dx] \\ &= f(x^{-1}(t - t_0; y_1), x^{-1}(t - t_0; y_2), \dots, x^{-1}(t - t_0; y_N))\end{aligned}$$

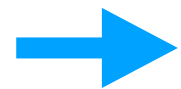
where  $x^{-1}(t - t_0; y)$  represents the inversion of the equation  $y = x(t - t_0; x_0)$

For arbitrary distribution function,

$$\begin{aligned} \int \langle [x], t_0 | \partial_{t_0} | f(t_0) \rangle [dx] &= \int \sum_{a=1}^N \left( X^i(x_a) \frac{\partial f[x]}{\partial x_a^i} \right) [dx] \\ &= - \int \sum_{a=1}^N \left( \partial_i X^i(x_a) f[x] \right) [dx] = 0. \end{aligned}$$

Equivalently,  $\frac{d}{dt} \int \langle [x], t | [dx] = 0 = \frac{d}{dt} \int |[x], t \rangle [dx]$

conservation law for the classical probability



$$\int \langle [x], t | | f(t_0) \rangle [dx] = 1$$

for *arbitrary*  $t$  and  $t_0$ .

More compactly,  $\langle Z | f(t_0) \rangle = 1$

$$\langle Z | \equiv \int \langle [x], t | [dx] \equiv \int \langle [x] [dx], \quad |Z \rangle \equiv \int |[x], t \rangle [dx] \equiv \int \langle [x] | [dx]$$

A statistical ensemble is said to be stationary when  $\hat{\mathcal{H}}_0|f(t_0)\rangle = 0$

This is satisfied simply if the distribution function has coordinate dependence only through conserved quantities:

$$f^{(0)}[x] \equiv g(H(x_1), H(x_2), \dots, H(x_n); K(x_1), K(x_2), \dots, K(x_n))$$

since  $X^i(x_a)\partial_{x_a^i}f^{(0)}[x] = 0$

Now, it should be clear that, as long as we remain within the realm of the usual Nambu equations of motion governed by the Hamiltonian  $\hat{\mathcal{H}}_0$ , there is no possibility of a *dynamical* mechanism for attaining equilibrium distributions

Thus, an important next issue is to what extent it is possible set up nontrivial kinds of interactions for Nambu particles consistently, such that an ensemble may reach an equilibrium statistical state, starting with a class of initial states that can be, most typically, a stationary statistical state with the distribution function  $f^{(0)}[x]$  of the above general form.





## **5. A solvable model of interaction**

### 5.1. *Dynamical evolution of classical statistical states as a Markov process*

$|F(t)\rangle$  time-dependent statistical  $N (\geq 2)$ -body state after including interaction

initial statistical state  $|F(0)\rangle = \int f^{(0)}[x] |[x], t_0\rangle [dx]$

#### *basic premise*

A real hermitian interaction Hamiltonian governs the Markov process toward equilibrium states.

$$|F(t)\rangle = \int F([x], t) |[x]\rangle [dx]$$

$$\partial_t |F(t)\rangle = \hat{\mathcal{H}}_I(t) |F(t)\rangle$$

interaction Hamiltonian  $\hat{\mathcal{H}}_I$   $\hat{\mathcal{H}}_I = \hat{\mathcal{H}}_I^\dagger$

time-dependent distribution function  $F([x], t) \equiv F(x_1, \dots, x_n, t)$

*microscopic reversibility*  $\langle [x] | \hat{\mathcal{H}}_I(t) | [y] \rangle = \langle [y] | \hat{\mathcal{H}}_I(t) | [x] \rangle$

## Interacting Nambu dynamics as a non-Hermitian Schrödinger dynamics

Equivalently, we could start from the “Schrödinger representation”

$$\partial_t |\tilde{F}(t)\rangle = -\hat{\mathcal{H}} |\tilde{F}(t)\rangle, \quad \hat{\mathcal{H}} \equiv \hat{\mathcal{H}}_0 - \hat{\mathcal{H}}_I$$

total ***non-hermitian Hamiltonian***

$$(\hat{\mathcal{H}}_0 - \hat{\mathcal{H}}_I)^\dagger = -(\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_I)$$

with **time-independent** field operators  $\psi(x, 0), \psi^\dagger(x, 0)$ .

$$|\tilde{F}(t)\rangle = e^{-\hat{\mathcal{H}}_0 t} |F(t)\rangle$$

Then  $|F(t)\rangle$  satisfies  $\partial_t |F(t)\rangle = \hat{\mathcal{H}}_I(t) |F(t)\rangle$

with **time-dependent** field operators in the “interaction representation”

$$\psi(x, t) = e^{\hat{\mathcal{H}}_0 t} \psi(x, 0) e^{-\hat{\mathcal{H}}_0 t}$$

$$\psi^\dagger(x, t) = e^{\hat{\mathcal{H}}_0 t} \psi^\dagger(x, 0) e^{-\hat{\mathcal{H}}_0 t}$$

N.B.: There is in general no time-inversion symmetry.

## 5.2. Requirements for the interaction Hamiltonian

For simplicity, only consider a 2-body interaction

$$\hat{\mathcal{H}}_I(t) = \frac{1}{4} \iiint \psi^\dagger(x_3, t) \psi^\dagger(x_4, t) \underline{V(x_3, x_4; x_1, x_2)} \psi(x_1, t) \psi(x_2, t) \prod_{a=1}^4 d^3x_a$$

symmetric under interchanges  $x_1 \leftrightarrow x_2$ ,  $x_3 \leftrightarrow x_4$ , and  $(x_1, x_2) \leftrightarrow (x_3, x_4)$

- ‘Interaction’ must necessarily be **non-local in phase space**, due to the first-order nature of the equations of motion.

(i) Nonnegativity: the off-diagonal matrix elements of  $\hat{\mathcal{H}}_I(t)$  must be *nonnegative*

$$|F(t + \Delta t)\rangle = e^{\Delta t \hat{\mathcal{H}}_I(t)} |F(t)\rangle \simeq (1 + \Delta t \hat{\mathcal{H}}_I(t)) |F(t)\rangle$$

↑  
**classical probability interpretation**

distribution functions cannot be negative at any time

(ii) Conservation of probability:

$$\langle Z | \hat{\mathcal{H}}_I(t) = 0 = \hat{\mathcal{H}}_I(t) | Z \rangle \quad \leftarrow \quad \langle Z | T(e^{\int_0^t \hat{\mathcal{H}}_I(t) dt}) | F(0) \rangle = 1 \quad \langle Z | \equiv \int \langle [x] [dx]$$

for arbitrary statistical state

(iii) Conservation of energies:

$$[\hat{\mathcal{H}}_I(t), \hat{H}] = 0 = [\hat{\mathcal{H}}_I(t), \hat{K}]$$



Then the kernel function must satisfy

$$(i) \quad V(x_1, x_2; x_3, x_4) \geq 0 \quad \text{for} \quad (x_1, x_2) \neq (x_3, x_4)$$

$$(ii) \quad \iint V(x_3, x_4; x_1, x_2) d^3x_3 d^3x_4 = 0 = \iint V(x_3, x_4; x_1, x_2) d^3x_1 d^3x_2$$

$$(iii) \quad \begin{aligned} V(x_3, x_4; x_1, x_2) (H(x_1) + H(x_2) - H(x_3) - H(x_4)) &= 0, \\ V(x_3, x_4; x_1, x_2) (K(x_1) + K(x_2) - K(x_3) - K(x_4)) &= 0. \end{aligned}$$

- An additional simplifying assumption:  $[\hat{\mathcal{H}}_I(t), \hat{\mathcal{H}}_0] = 0$

guaranteed if the coordinate dependence of the kernel is only through the energy functions

All of these conditions are compatible with the linear  $\mathcal{N}$ -symmetry  $\text{SL}(2, \mathbb{R})$

### 5.3. *The H-theorem for the evolution of statistical states*

Under the foregoing conditions, we can *rigorously* prove that

$$H(t + \Delta t) = \int h(\langle [x] | F(t + \Delta t) \rangle) [dx] \leq \int h(\langle [x] | F(t) \rangle) [dx] = H(t)$$

for the ‘H-function’

$$h(F(t)) = \langle [x] | F(t) \rangle \log \langle [x] | F(t) \rangle$$

*Essentially follows the old methods developed by Husimi (and Stueckelberg) in the 1940s in ordinary Hamiltonian dynamics*

$$\downarrow -H = S$$

“the principle of entropy”

equilibrium state  $|F(\infty)\rangle$ , satisfying  $\hat{\mathcal{H}}_I |F(\infty)\rangle = 0$ ,  
must minimize the H-function

The interaction can be called to be “***stochastic***”,

and the Markov process induced by the interaction is in general  
***irreversible, reflecting non-hermiticity of the dynamics***

#### 5.4. *Ansatz for stochastic interaction: an exactly solvable model*

$$V(x_3, x_4; x_1, x_2) \equiv g^2 (v(x_3, x_4; x_1, x_2) - i(x_3, x_4; x_1, x_2))$$

$$i(x_3, x_4; x_1, x_2) \equiv \frac{1}{2} (\delta^3(x_3 - x_1) \delta^3(x_4 - x_2) + \delta^3(x_3 - x_2) \delta^3(x_4 - x_1))$$

$$v(x_3, x_4; x_1, x_2) = \bar{v}(x_3, x_4; x_1, x_2) \\ \times \delta(H(x_1) + H(x_2) - H(x_3) - H(x_4)) \delta(K(x_1) + K(x_2) - K(x_3) - K(x_4))$$

$$\bar{v}(x_3, x_4; x_1, x_2) = [\rho_{H,K}(H(x_3) + H(x_4), K(x_3) + K(x_4)) \rho_{H,K}(H(x_1) + H(x_2), K(x_1) + K(x_2))]^{-1/2}$$

$$\rho_{H,K}(h, k) \equiv \iint \delta(H(x_1) + H(x_2) - h) \delta(K(x_1) + K(x_2) - k) d^3x_1 d^3x_2$$

the volume of a compact 4D object embedded in 6D coordinate space  $(x_1, x_2)$

with the constraints  $H(x_1) + H(x_2) = h$ ,  $K(x_1) + K(x_2) = k$

### 5.5. Case of quadratic energy functions

$$H = A_1(x^1)^2 + A_2(x^2)^2 + A_3(x^3)^2$$

$$K = B_1(x^1)^2 + B_2(x^2)^2 + B_3(x^3)^2$$

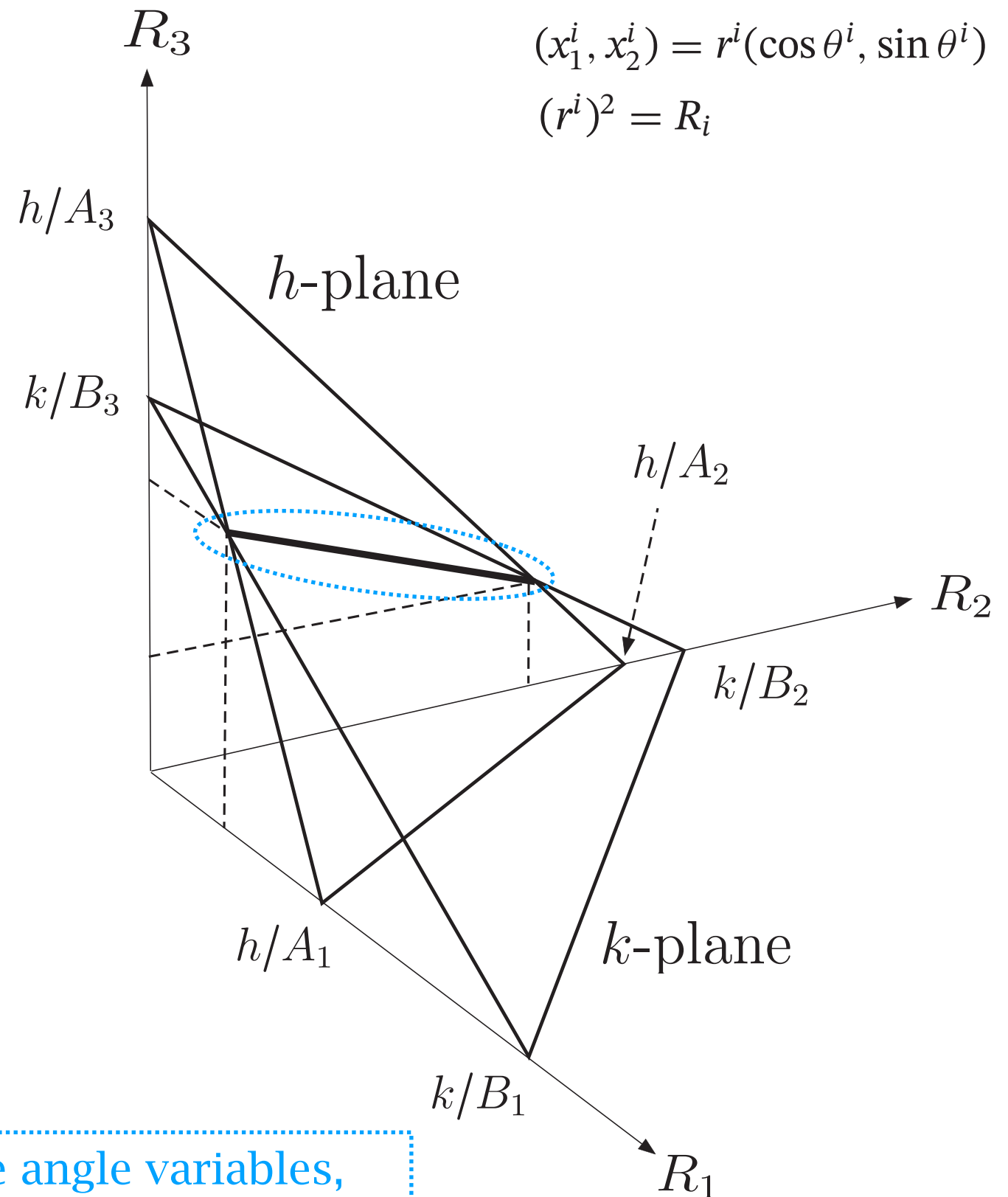
$$\rho_{H,K}(h, k) = \frac{\pi^3 |hB_3 - kA_3|}{|A_3B_1 - A_1B_3| |A_3B_2 - A_2B_3|}$$

$\mathcal{N}$ -symmetry

$$\begin{pmatrix} H' \\ K' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H \\ K \end{pmatrix}, \quad ad - bc = 1$$

$$A'_1 B'_2 - A'_2 B'_1 = A_1 B_2 - A_2 B_1 \quad \text{etc}$$

Upon integration over the angle variables,  
the intersecting 4D object reduces to an  
intersecting line.



5.6. *The exact spectrum and eigenfunctions for the functional operator  $v(x_3, x_4; x_1, x_2)$*

Consider the integral equation associated with the kernel function

$$\iint v(x_3, x_4; x_1, x_2) f_\lambda(x_1, x_2) d^3x_1 d^3x_2 = \underline{\lambda} f_\lambda(x_3, x_4)$$

Can prove that

- **eigenvalues** consist of only 1 and 0 with infinite degeneracy for each case.

$$f_1(x_1, x_2|k, h) = f(k, h)$$

$$\iint f_0(x_1, x_2|h, k) d^3x_1 d^3x_2 = 0$$

due to the following projector property of the kernel function:

$$\iint v(x_3, x_4; y_1, y_2) v(y_1, y_2; x_1, x_2) d^3y_1 d^3y_2 = v(x_3, x_4; x_1, x_2)$$

Thus the eigenvalue spectrum of the total kernel function

$$V(x_3, x_4; x_1, x_2) = g^2(v(x_3, x_4; x_1, x_2) - i(x_3, x_4; x_1, x_2))$$

is  $g^2 - g^2 = 0$  and  $0 - g^2 = -g^2$

**coordinate dependence of the eigenfunctions for eigenvalue 0 is allowed only through the sums of energies.**



## An analog-matrix model for the kernel function $v(x_3, x_4; x_1, x_2)$

Consider an  $n \times n$  “positive” matrix  $p_{ij}^{(n,n)}$  with entries  $p_{ij}^{(n,n)} = p_{ji}^{(n)} = 1/n$ , satisfying

$$\sum_k p_{ik}^{(n,n)} p_{kj}^{(n,n)} = p_{ij}^{(n,n)}, \quad \sum_i p_{ij}^{(n,n)} = 1 = \sum_i p_{ij}^{(n,n)},$$

→  $\sum_j p_{ij}^{(n,n)} r_j = n \times 1/n = 1.$

$$\begin{pmatrix} p^{(2,2)} & 0^{(2,3)} & 0^{(2,4)} & \cdot & \cdot & \dots \\ 0^{(3,2)} & p^{(3,3)} & 0^{(3,4)} & \cdot & \cdot & \dots \\ 0^{(4,2)} & 0^{(4,3)} & p^{(4,4)} & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \dots & \cdot & \cdot & \dots \end{pmatrix}$$

eigenvectors with eigenvalue 1

$$r_i^{(n)} = 1 \text{ for all } i = 1, \dots, n$$

$$\begin{pmatrix} 0^{(2)} \\ 0^{(3)} \\ \cdot \\ \cdot \\ 0^{(n-1)} \\ 1^{(n)} \\ 0^{(n+1)} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

eigenvalue 0

$$\sum_i v_i^{(n)} = 0$$

$$\begin{pmatrix} 0^{(2)} \\ 0^{(3)} \\ \cdot \\ \cdot \\ 0^{(n-1)} \\ v^{(n)} \\ 0^{(n+1)} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$



## 6. Equilibrium statistical states

## 6.1. Approaches to equilibrium statistical states

The interaction Hamiltonian acts on a generic N-body basis state  $|[x]\rangle$  as

$$\mathcal{H}_I|[x]\rangle = \frac{1}{4\sqrt{N!}} \iint V(x'_3, x'_4; x_a, x_b) \psi^\dagger(x'_3, 0) \psi^\dagger(x'_4, 0) \sum_{a,b (a \neq b)}^N \left( \prod_{c \neq a,b}^N \psi^\dagger(x_c, 0) \right) |0\rangle d^3x'_3 d^3x'_4$$

generic N-body statistical state  $|F(t)\rangle = \int F([x], t) |[x]\rangle [dx]$

$$F([x], t) = F(x_1, \dots, x_N, t) \quad \rightarrow \quad \sum_{a,b (a \neq b)}^N \iint V(x'_3, x'_4; x_a, x_b) F(x_1, \dots, x_N, t) d^3x_a d^3x_b$$

**The highest eigenvalue 0 corresponding to equilibrium states, satisfying**

$$\lim_{t \rightarrow \infty} \hat{\mathcal{H}}_I |F(t)\rangle = 0$$

is realized if and only if  $F([x], \infty)$  depends on the coordinates only through the total energies,

$$\sum_{a=1}^N K(x_a) \text{ and } \sum_{a=1}^N H(x_a)$$

since the coordinate dependence of the eigenfunction is allowed only through

$H(x_a) + H(x_b)$  and  $K(x_a) + K(x_b)$  for all possible pairs  $(x_a, x_b)$ .

infinitely degenerate  $f \left( \sum_{a=1}^N K(x_a), \sum_{a=1}^N H(x_a) \right)$

The next possible eigenvalue corresponding to the first excited states is

$$-\frac{g^2}{4}(N(N-1) - (N-1)(N-2)) = -\frac{1}{2}g^2(N-1)$$

difference between the numbers of pairs  
of 2-body states with eigenvalue 0



Approach to an equilibrium state obeys in general an exponential law

$$e^{-g^2(N-1)t/2}$$

for sufficiently large  $t$  for any  $N \geq 2$

Infinite degeneracies of these eigenstates owe essentially to the linear nature of the equations describing the Markov process governed by  $\hat{\mathcal{H}}_I$

More refined characterization of equilibrium states will be attained only when specific conditions for the initial statistical states are given.

It is possible to take into account such conditions by applying the H-theorem:

## 6.2. *Equilibrium statistical states from the H-theorem*

$H = \int F_\infty[x] \log F_\infty[x][dx]$  is minimum under given initial conditions

$$F_\infty[x] \equiv \langle [x] | F(\infty) \rangle$$

We here only give two typical equilibrium states which are obtained by standard minimization procedure.

| Equilibrium statistical states                                                                                                                                                                                                                                                                                                                                                                              | given conditions                                                                                   |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------|
| <div style="border: 1px solid #008080; padding: 5px; display: inline-block;"> <i>Generalized microcanonical states</i> </div> $F_\infty[x] = e^{-\alpha_0(h_0, k_0)} \delta\left(\sum_{a=1}^N H(x_a) - h_0\right) \delta\left(\sum_{a=1}^N K(x_a) - k_0\right)$ $e^{\alpha_0(h_0, k_0)} = \iint \cdots \int \delta\left(\sum_{a=1}^N H(x_a) - h_0\right) \delta\left(\sum_{a=1}^N K(x_a) - k_0\right) [dx]$ | <p>exact values of total energies</p> $\sum_{a=1}^N H(x_a) = h_0, \quad \sum_{a=1}^N K(x_a) = k_0$ |



Equilibrium statistical states

given conditions

*Generalized canonical states*

$$F_{\infty}[x] = \exp \left\{ -\alpha - \beta_h \sum_{a=1}^N H(x_a) - \beta_k \sum_{a=1}^N K(x_a) \right\}$$

$$e^{\alpha} = \int \exp \left\{ -\beta_h \sum_{a=1}^N H(x_a) - \beta_k \sum_{a=1}^N K(x_a) \right\} [dx]$$

$$h = \int \sum_{a=1}^N H(x_a) \exp \left\{ -\alpha - \beta_h \sum_{a=1}^N H(x_a) - \beta_k \sum_{a=1}^N K(x_a) \right\} [dx]$$

$$k = \int \sum_{a=1}^N K(x_a) \exp \left\{ -\alpha - \beta_h \sum_{a=1}^N H(x_a) - \beta_k \sum_{a=1}^N K(x_a) \right\} [dx]$$

$$\rightarrow |F(\infty)\rangle_{(\beta_h, \beta_k)} \equiv \frac{1}{\sqrt{N!}} \Psi^{\dagger}(\beta_h, \beta_k)^N |0\rangle$$

$$\Psi^{\dagger}(\beta_h, \beta_k) = \frac{\int e^{-\beta_h H(x) - \beta_k K(x)} \psi^{\dagger}(x, t_0) d^3x}{\int e^{-\beta_h H(x) - \beta_k K(x)} d^3x}$$

average values of total energies

$$\int F_{\infty}[x] \sum_{a=1}^N H(x_a) [dx] = h$$

$$\int F_{\infty}[x] \sum_{a=1}^N K(x_a) [dx] = k$$

*As anticipated by Nambu, canonical equilibrium states are characterized by two independent ‘temperatures’:*

***However, there is a crucial difference from the usual temperatures !***

## $\mathcal{N}$ -symmetry for equilibrium statistical states

(A) The microcanonical distribution is invariant under  $\begin{pmatrix} H' \\ K' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H \\ K \end{pmatrix}, \quad ad - bc = 1$

$$\begin{pmatrix} h'_0 \\ k'_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h_0 \\ k_0 \end{pmatrix}$$

$$\delta\left(\sum_{a=1}^N H(x_a) - h_0\right) \delta\left(\sum_{a=1}^N K(x_a) - k_0\right) = \delta\left(\sum_{a=1}^N H'(x_a) - h'_0\right) \delta\left(\sum_{a=1}^N K'(x_a) - k'_0\right)$$

(B) The generalized canonical distribution is invariant

$$\begin{pmatrix} \beta'_h & \beta'_k \end{pmatrix} = \begin{pmatrix} \beta_h & \beta_k \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \beta'_h H'(x) + \beta'_k K'(x) = \beta_h H(x) + \beta_k K(x)$$

→ “relativity” of temperatures.

The temperatures have no absolute meaning in Nambu dynamics

$$(\beta_h, \beta_k) = (0, 1)g \quad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \text{SL}(2, \mathbb{R})$$

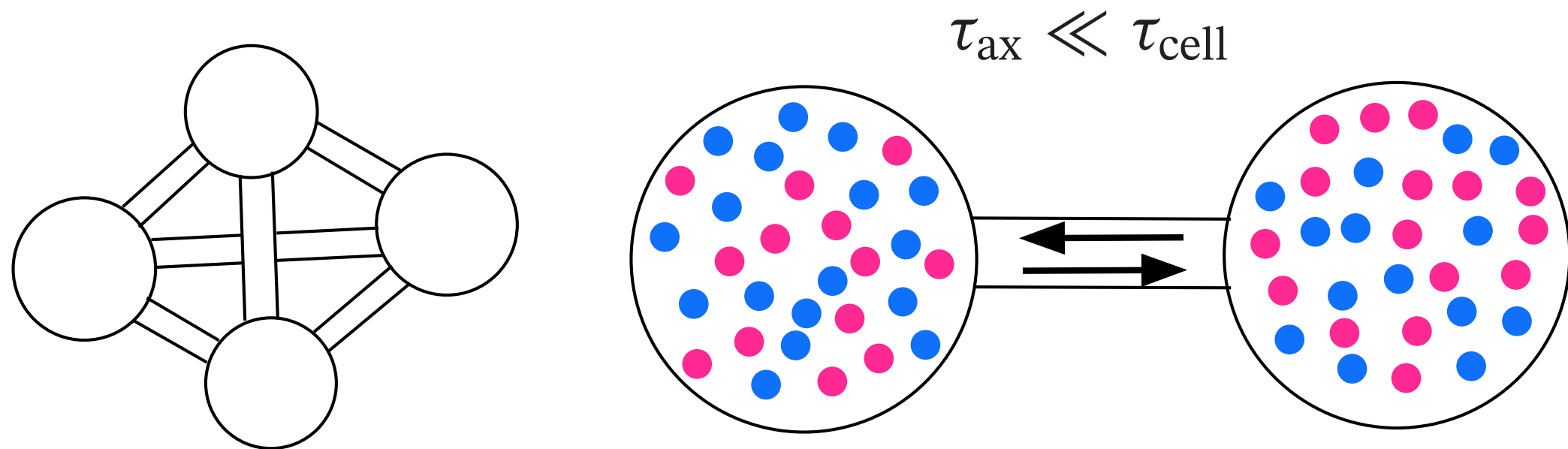


## **7. Concluding remarks**

## *Summary of our results*

- Investigated how Nambu's conjecture for generalized statistical mechanics is realized concretely from a dynamical viewpoint: This necessarily requires us to include self-interaction which is non-local in phase space, which has never been discussed in earlier studies.
- Proposed a toy model of interaction, which is exactly solvable and universally applicable for any specific forms of the multiple Hamiltonians.
- Clarified crucial properties of Nambu dynamics with respect to the  $SL(2, \mathbb{R})$  symmetry, especially on the nature of temperatures.

*How is the nonlocal interaction realizable? A speculative network system*



**Fig. 2.** Left: A network of virtual cells connected through axons ( $N = 4$ ). Right: Exchange of two kinds of molecules between a pair of cells through an axon connecting them. Blue or red molecules only carry energy  $H$  or  $K$ , respectively.

More complicated kernel function for interaction would yield much more rich phenomena in approaching to equilibrium, depending on the choice of the initial statistical state.



## *Possible applicability ?*

Though I cannot suggest any immediate application of the present work right now, interesting new features as above have been found, compared with the ordinary statistical mechanics based on the usual Hamiltonian dynamics.

Is this just a mathematical curiosity?      May be or not . ....

**Hopefully, it might be useful in unexpected ways in the future.....**

*just like string theory has been in the early 1970s*

「弦理論とは何か一言で述べよ」と問われたとすると、私は「弦理論は量子重力理論として紫外問題の根本的解決とゲージ相互作用との統一を同時に成し遂げられるほぼ唯一のアプローチである」と答える。弦理論は半世紀を超える歴史にも拘らず依然として完成した理論ではなく、これまでの展開の過程で多くの新たな驚くべき帰結や視点をもたらし、場の理論と量子重力の理解を深めるのに貢献してきた。現在でもこうした状況は続いており、その真価の解明は今世紀の大きな課題であると言って過言ではない。この観点から、弦理論誕生の歴史的背景・経緯、弦による量子重力に関する私自身の仕事とその後の進展を振り返り、弦理論の意味を考える。

## 1. 弦理論の萌芽と源流、そして双対共鳴模型へ

歴史を辿ると、基本的力学自由度としての弦の物理的描像の萌芽をディラックの 1950~60 年代の仕事<sup>1)</sup>に見出せる。量子電気力学のゲージ不変な定式化を論じた中で、基本的自由度として場の代わりにファラデーの電気力線を採用し、光子は閉じた電気力線、電子は開いた電気力線の端点として扱うのが合理的だと述べた。現在の弦理論では、開弦の端点は‘D-ブレーン’ (図 1 右) でそこを伝播するゲージ場が生成される一方、閉弦はバルク時空を伝播する重力子を生み出す。また、彼は広がった自由度の相対論的量子力学に向けて、拘束条件付きの正準形式の一般論を展開した。それを応用した閉じた 2 次元面が時空で成す 3 次元世界体積作用関数に基づく広がった電子模型 (1962 年) は、弦理論の拡張としての M 理論の萌芽と言える。当然、一般論の同様な応用として 2 次元世界面の作用原理による弦の力学の正準形式にも直ちに至ることが出来る。このようにディラックは弦の量子力学の要となる考え方と方法論を 50 年代にすでに手にしており、弦理論への道を切り拓いていた。

一方、弦理論を生み出した S 行列理論の源流を引き起したのはハイゼンベルクである。彼は 1940 年代前半の一連の論文<sup>2)</sup>で「今後の理論の変更によって影響を受けないような、将来の理論でも必須となる概念を波動場の量子論から抽出する」と述べ、S-行列理論の思想を結晶化すると同時に、局所場理論の限界を特徴づける新たな自然定数としての‘基本的長さ’の重要性を強調した。粒子散乱の始状態と終状態間の遷移確率という最終観測量だけを扱う S 行列の

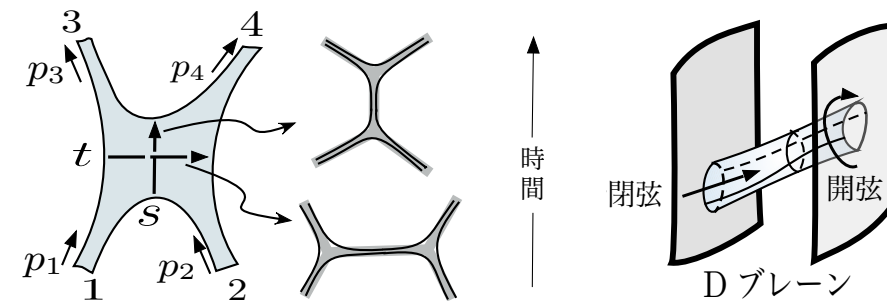


図 1: チャンネル双対性 (左) と開弦/閉弦双対性 (右)

関数  $\alpha(t)$  (一般には複素数) をレッジエ軌跡と呼ぶ。これは変数  $t$  に関する極特異点に対応する複合粒子の交換により散乱を記述するので、 $t$  チャンネル描像と呼ぼう。また、ハドロンが衝突して一般には別の複合粒子に対応する共鳴状態ができ、それが複数の複合粒子に崩壊して散乱が起きる‘共鳴’描像も現象論的に有効である。その振幅は  $s = (p_1 + p_2)^2$  に関する極を持つので、 $s$  チャンネル描像と呼ぼう。

そこから新たな指導原理として、60 年代終盤、どちらの描像に基づいても解析的に同一の S 行列が得られるという考えが浮かび上がった。この要請 (以下、チャンネル双対性と呼ぶ) は理論に対する極めて強い‘自己縛り’ (bootstrap) になり得る。これを鮮やかに具現化した散乱振幅の公式

$$V(s, t) = g^2 \int_0^1 x^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1} dx \quad (1)$$

がヴェネツィアノ (G. Veneziano) により発見された<sup>5)</sup>。レッジエ軌跡は実 1 次式  $\alpha(s) \equiv \alpha' s + \alpha(0)$ ,  $g$  はハドロンの 3 点相互作用の強さを表す結合定数である。この式そのもの

Perhaps useful to extend our perspective beyond physical systems to the area of **biochemical systems or even to economical or sociological systems**, which are ubiquitously abundant with various complicated networks.

It would be exciting if we could apply the method of quantum field theory with similar sorts of nonlocal interactions as ours to such areas.