

Exact Lattice super Yang-Mills の新しい時空構造とアルジェブラ

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In collaboration with K. Nagata and J. Saito

A. D'Adda, N.K., K. Nagata and J. Saito,
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A. D'Adda, I. Kanamori N.K. and K. Nagata,
Phys. Lett.B 633(2006)645-652

Gauge Theory on the Random Lattice $\longrightarrow$ Supergravity ?

Form

Simplex

Gauge Theory + Gravity ?

0	ϕ	
1	$\omega = \omega_\mu dx^\mu$	
2	$B = B_{\mu\nu} dx^\mu dx^\nu$	
3	$\Omega = \Omega_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho$	
$\vdots$	$\vdots$	$\vdots$

$$\mathcal{A} = \mathbf{j}(\overset{1}{\omega} + \overset{3}{\Omega} + \dots) + \mathbf{k}(\overset{0}{\phi} + \overset{2}{B} + \dots) \\ + \mathbf{i}(\chi^{(1)} + \chi^{(3)} + \dots) + \mathbf{i}(\chi^{(0)} + \chi^{(2)} + \dots)$$

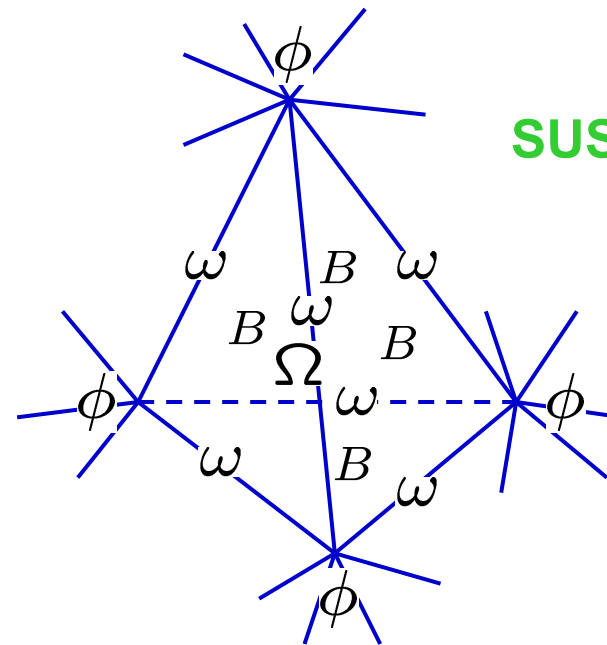
(quaternion based formulation: **origin ?**)

$$S = \int \text{Tr}(\frac{1}{2} \mathcal{A} Q \mathcal{A} + \frac{1}{3} \mathcal{A}^3)$$

N.K.&Watabiki '92

Boson $\longleftrightarrow$ Fermion ?

SUSY ?



3d Chern-Simons gravity by Witten (1988/89)

There are two main obstacles for exact lattice SUSY.

- 1) Breakdown of Leibniz rule for difference operator
- 2) Chiral fermion doublers

$$(\Delta_+ \phi_i(x) = \phi_i(x + a) - \phi_i(x))$$

Consider the simplest SUSY algebra in 1-dim.

$$\{Q, Q\} = 2H = 2i\partial \rightarrow 2i\Delta_+ \longrightarrow Q^2 = i\Delta_+$$

$$\mathbf{1)} \quad \Delta_+(\phi_1(x)\phi_2(x)) = (\Delta_+\phi_1(x))\phi_2(x) + \phi_1(x+a)(\Delta_+\phi_2(x))$$

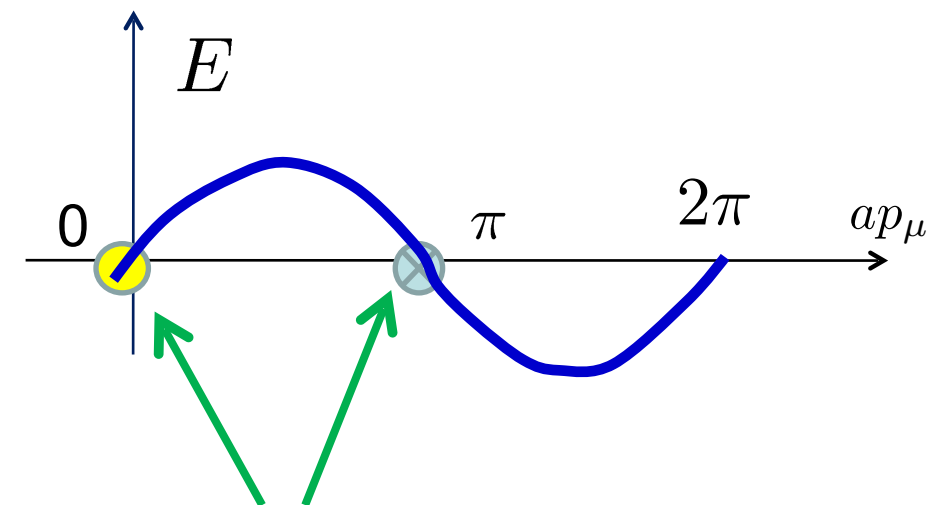
$$Q(\phi_1(x)\phi_2(x)) = (Q\phi_1(x))\phi_2(x) + \phi_1(x)(Q\phi_2(x))$$

2) chiral fermion doublers $\longrightarrow$ **#fermions $\neq$ #bosons** SUSY breaking

$$\bar{\psi}(x) i \gamma_{\mu} \{ \psi(x + a \hat{\mu}) - \psi(x - a \hat{\mu}) \} / 2a$$

$$\frac{1}{\gamma^{\mu} \sin a p_{\mu}}, \quad a p_{\mu} = 0, \pi$$

$D=N=2$



doubling of fermions

$$\# \text{ doublers} = 2^D \quad (D = \text{dimension})$$

$$= 2^N \quad (\text{d.o.f. of } N = \text{extended SUSY})$$

$D=N=2$
 $D=N=4$ extended SUSY

doubler fermions

$$\psi(p), \quad \psi\left(p + \frac{\pi}{a}\right)$$

We try to solve the two difficulties
1) and 2) at the same time.



Link approach

$$Q^2 = i\Delta_+$$

$$\Delta_+(\phi_1(x)\phi_2(x)) = (\Delta_+\phi_1(x))\phi_2(x) + \phi_1(x+a)\Delta_+\phi_2(x)$$

Ansatz: $Q(\phi_1(x)\phi_2(x)) = \{Q\phi_1(x)\}\phi_2(x) + \underbrace{\phi_1(x+a_Q)}_{\text{shift parameter}}Q\phi_2(x)$

We introduce similar shift for Q-operation !

$$\begin{aligned} Q^2(\phi_1(x)\phi_2(x)) &= \{Q^2\phi_1(x)\}\phi_2(x) - \{Q\phi_1(x+a_Q)\}Q\phi_2(x) \\ &\quad + \{Q\phi_1(x+a_Q)\}Q\phi_2(x) + \phi_1(x+2a_Q)Q^2\phi_2(x) \\ &= \{Q^2\phi_1(x)\}\phi_2(x) + \phi_1(x+2a_Q)Q^2\phi_2(x) \end{aligned}$$

$$Q^2 = i\Delta_+ \leftrightarrow 2a_Q = a$$

Geometrical interpretation of link approach

shift parameter of Q

$$Q(\phi_1(x)\phi_2(x)) = \{Q\phi_1(x)\}\phi_2(x) + \phi_1(x + a_Q)Q\phi_2(x) \quad a_Q = a/2$$

$$[Q(\phi_1(x)\phi_2(x))]_{x+a/2,x} = \underbrace{\{Q\phi_1(x)\}_{x+a/2,x}}_{\text{fermionic link}} [\phi_2(x)]_{x,x} + [\phi_1(x + a/2)]_{x+a/2,x+a/2} \{Q\phi_2(x)\}_{x+a/2,x}$$

coordinate ordering rule

Super charge operation generates a fermionic link hole.

$$Q^2 = i\Delta_+$$

Q is $a/2$ translation generator

Δ_+ is a translation generator

Dirac-Kahler Twisting Mechanism

of chiral fermion doublers = 2^D = # of independent γ matrices
 N extended SUSY d.o.f. 2^N

★ Use the chiral fermion doublers d.o.f. as super multiplets.

$$D=N=2: \quad Q_{\alpha i} = (\mathbf{1}Q + \gamma^\mu Q_\mu + \gamma^1 \gamma^2 \tilde{Q})_{\alpha i} \quad (\alpha, i = 1, 2)$$

$$\{Q_{\alpha i}, \bar{Q}_{j\beta}\} = 2\delta_{ij}(\gamma)_{\alpha\beta} i\Delta_{\pm\mu} \longrightarrow \begin{cases} \{Q, Q_\mu\} = i\Delta_{+\mu} \\ \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu} \\ \{\text{others}\} = 0 \end{cases}$$

$$\{Q_A, Q_B\} = -f_{AB}^\mu \Delta_{\pm\mu}$$

D=N=2 Dirac-Kahler twisted SUSY algebra

$$\begin{aligned} Q^2 &= i\Delta_+ \\ 2a_Q &= a \end{aligned} \quad \begin{array}{c} \uparrow \\ \text{lattice constant} \end{array}$$

$$a + a_\mu = n_\mu, \quad \tilde{a} + a_\mu = -|\epsilon_{\mu\nu}|n_\nu$$

$$\begin{aligned} a + \tilde{a} + a_1 + a_2 &= 0 & (a_A + a_B = n_\mu) \\ \uparrow & \\ \text{one free shift vector !} & \end{aligned}$$

super charge

covariant derivative

$$\{Q_A, Q_B\} = -f_{AB}^\mu \Delta_{\pm\mu} \longrightarrow \{D_A, D_B\} = f_{AB}^\mu \Delta_{\pm\mu}$$

Introduction of gauge symmetry

continuum



$$\begin{aligned} \nabla_{\underline{\mu}} &= \partial_{\underline{\mu}} - i\Gamma_{\underline{\mu}}(x, \theta_A, \theta_B) & \{\nabla_A, \nabla_B\} &= f_{AB}^\mu \nabla_{\underline{\mu}} \\ &= \partial_{\underline{\mu}} - iA_{\underline{\mu}} + \dots \end{aligned}$$

$$e^{\partial_{\underline{\mu}}} \phi(x) = \phi(x + n_{\underline{\mu}})$$

$$[e^{\partial_{\underline{\mu}}}, \phi(x)] = \phi(x + n_{\underline{\mu}}) - \phi(x) = \Delta_{+\underline{\mu}} \phi(x)$$

lattice derivative $\longrightarrow$
exponentialize continuum

$$\{\nabla_A, \nabla_B\} = -f_{AB}^\mu e^{-\nabla_{\underline{\mu}}}$$

$$\{\nabla_A, \nabla_B\} = f_{AB}^\mu e^{\nabla_{\underline{\mu}}}$$

$$\{\nabla_A, \nabla_B\} = -f_{AB}^\mu e^{-\nabla_\mu} \quad \nabla_\mu = \partial_\mu - i\Gamma_\mu(x, \theta_A, \theta_B)$$

$e^{\pm \nabla_\mu}$ is defined on a site because

$$\nabla_\mu \rightarrow \mathcal{G}^{-1}(x) \nabla_\mu \mathcal{G}(x) \quad \leftarrow \text{gauge trans. of covariant deriv.}$$

$$e^{\mp \nabla_\mu} \rightarrow \mathcal{G}^{-1}(x) e^{\mp \nabla_\mu} \mathcal{G}(x)$$

By using Trotter formula, we can show

$$e^{\mp \nabla_\mu} = e^{\mp \partial_\mu} e^{\pm \partial_\mu} e^{\mp \partial_\mu \pm i\Gamma_\mu(x)}$$

$$= e^{\mp \partial_\mu} \mathcal{P} \exp \left[\pm i \int_x^{x \pm n_\mu} \Gamma_\mu(s) ds^\mu \right] \rightarrow e^{\mp \partial_\mu} (\mathcal{U}_{\pm\mu})_{x \pm n_\mu, x}$$

$\lambda|$

$$e^{\pm i\Gamma_\mu(x)} \simeq e^{\pm iA_\mu(x) + \dots}$$

$$\{\nabla_A(x), \nabla_B(x)\} = -f_{AB}^\mu e^{-\partial_\mu} (\mathcal{U}_{+\mu})_{x+n_\mu, x}$$

$$\{\nabla_A(x), \nabla_B(x)\} = +f_{AB}^\mu e^{+\partial_\mu} (\mathcal{U}_{-\mu})_{x-n_\mu, x}$$

$$e^{\partial_\mu} \varphi(x) = \varphi(x + n_\mu), \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad (n_\mu)^\nu = \delta_\mu^\nu, \quad n_\mu \cdot \partial = \partial_\mu$$

$$e^{a_A \cdot \partial} \varphi(x) = \varphi(x + a_A), \quad a_A \cdot \partial = (a_A)^\nu \partial_\nu$$

$$a_A \cdot \partial + a_B \cdot \partial = n_\mu \cdot \partial = \partial_\mu \quad (a_A + a_B = n_\mu)$$

link generators:

$$[e^{\partial_\mu}, \varphi(x)] = \varphi(x + n_\mu) - \varphi(x)$$

$$[e^{a_A \cdot \partial}, \varphi(x)] = \varphi(x + a_A) - \varphi(x)$$

$$a_A \cdot \partial + a_B \cdot \partial = n_\mu \cdot \partial = \partial_\mu \quad (a_A + a_B = n_\mu)$$

$$\{\nabla_A(x), \nabla_B(x)\} = -f_{AB}^\mu e^{-\partial_\mu} (\mathcal{U}_{+\mu})_{x+n_\mu, x}$$

$$e^{\partial_\mu} \{\nabla_A(x), \nabla_B(x)\} = -f_{AB}^\mu (\mathcal{U}_{+\mu})_{x+n_\mu, x}$$

$$|| \quad (a_A + a_B = n_\mu, \quad a_A \cdot \partial + a_B \cdot \partial = \partial_\mu)$$

$$\begin{aligned} & \underline{e^{+a_A \cdot \partial} e^{+a_B \cdot \partial} \nabla_A(x) e^{-a_B \cdot \partial} e^{+a_B \cdot \partial} \nabla_B(x) + e^{+a_B \cdot \partial} e^{+a_A \cdot \partial} \nabla_B(x) e^{-a_A \cdot \partial} e^{+a_A \cdot \partial} \nabla_A(x)} \\ &= \underline{e^{+a_A \cdot \partial} \nabla_A(x + a_B) e^{+a_B \cdot \partial} \nabla_B(x) + e^{+a_B \cdot \partial} \nabla_B(x + a_A) e^{+a_A \cdot \partial} \nabla_A(x)} \end{aligned}$$

$$(\nabla_A)_{x+a_A, x} \equiv e^{+a_A \cdot \partial} \nabla_A(x), \quad (\nabla_B)_{x+a_B, x} \equiv e^{+a_B \cdot \partial} \nabla_B(x)$$

$$\{\nabla_A, \nabla_B\}_{x+a_A+a_B, x} = -f_{AB}^\mu (\mathcal{U}_{+\mu})_{x+n_\mu, x}$$

$$\{\nabla_A, \nabla_B\}_{x+a_A+a_B, x} \equiv (\nabla_A)_{x+a_A+a_B, x+a_B} (\nabla_B)_{x+a_B, x} + (\nabla_B)_{x+a_B+a_A, x+a_A} (\nabla_A)_{x+a_A, x}$$

Fermion fields & SUSY transformation

$$\begin{aligned}\{\nabla, \nabla_\mu\}_{x+a+a_\mu, x} &= i(\mathcal{U}_{+\mu})_{x+n_\mu, x}, \\ \{\tilde{\nabla}, \nabla_\mu\}_{x+\tilde{a}+a_\mu, x} &= i\epsilon_{\mu\nu}(\mathcal{U}_{-\nu})_{x-n_\nu, x}, \\ \{\text{others}\} &= 0\end{aligned}$$

definition of fermions

$$[\nabla_A, \mathcal{U}_{\pm\mu}] \sim [\nabla_A, \{\nabla_B, \nabla_C\}] \sim \text{fermion}(\rho, \tilde{\rho}, \lambda_\mu) \text{ or } 0$$

Graded Jacobi identities (D=N=2)

$$[\nabla_A, \{\nabla_B, \nabla_C\}] + [\nabla_B, \{\nabla_C, \nabla_A\}] + [\nabla_C, \{\nabla_A, \nabla_B\}] = 0$$

$$[\nabla_A, \text{fermion}(\rho, \tilde{\rho}, \lambda_\mu)] \sim \{\nabla_A, [\nabla_B, \{\nabla_C, \nabla_D\}]\} \sim [\mathcal{U}, \mathcal{U}] + K \text{ or } 0$$

↙ auxiliary field

$$s_A \varphi = [\nabla_A, \varphi]$$

SUSY transformation of $\mathcal{U}_{\pm\mu}, \rho, \tilde{\rho}, \lambda_\mu, K$
can be obtained by graded Jacobi identities

$$s \longleftrightarrow \nabla \longleftrightarrow a$$

$$\tilde{s} \longleftrightarrow \tilde{\nabla} \longleftrightarrow \tilde{a}$$

$$s_\mu \longleftrightarrow \nabla_\mu \longleftrightarrow a_\mu$$

SUSY transformation of N=2 super multiplets

$$s_A \varphi = [\nabla_A, \varphi]$$

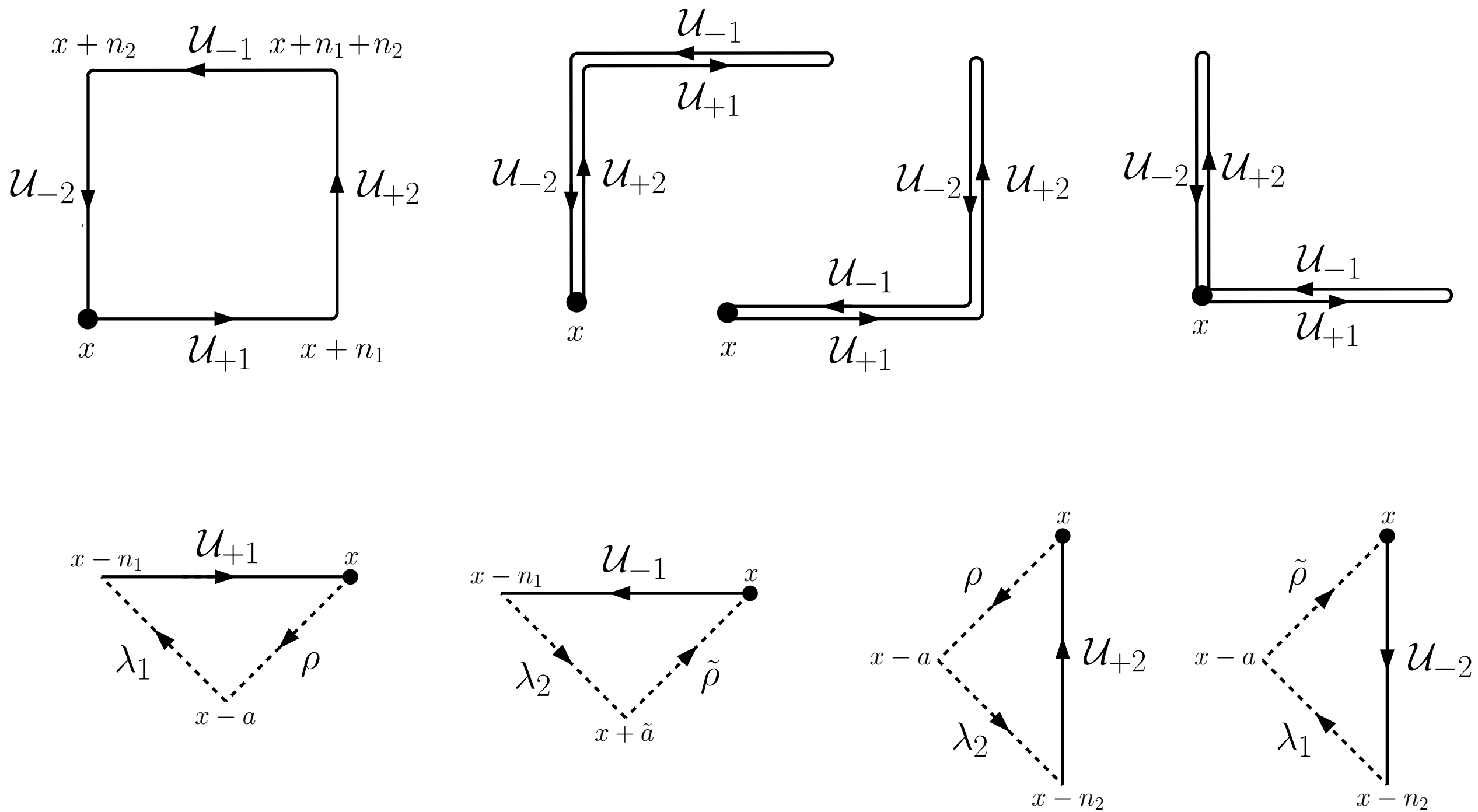
	s	$\tilde{s}$	s_μ
$\mathcal{U}_{+\nu}$	0	$+\epsilon_{\nu\rho}\lambda_\rho$	$-\epsilon_{\mu\nu}\tilde{\rho}$
$\mathcal{U}_{-\nu}$	$-\lambda_\nu$	0	$-\delta_{\mu\nu}\rho$
λ_ν	0	0	$-i[\mathcal{U}_{+\mu}, \mathcal{U}_{-\nu}] + \delta_{\mu\nu}(K + \frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}])$
ρ	$-\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	$+\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \mathcal{U}_{-\sigma}]$	0
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho}, \mathcal{U}_{+\sigma}]$	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	0
K	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \lambda_\rho]$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \lambda_\sigma]$	$-\frac{i}{2}[\mathcal{U}_{+\mu}, \rho] - \frac{i}{2}\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \tilde{\rho}]$

N=D=2 twisted SYM

$$S \equiv \frac{1}{4} \sum_x \text{Tr } s \tilde{s} \epsilon_{\mu\nu} s_\mu s_\nu \mathcal{U}_{+\mu} \mathcal{U}_{-\mu}$$

$$= \sum_x \text{Tr} \left[\frac{1}{4} [\mathcal{U}_{+\mu}, \mathcal{U}_{-\mu}]_{x,x} [\mathcal{U}_{+\nu}, \mathcal{U}_{-\nu}]_{x,x} + K_{x,x}^2 \right. \\ \left. - \frac{1}{4} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} [\mathcal{U}_{+\mu}, \mathcal{U}_{+\nu}]_{x,x-n_\mu-n_\nu} [\mathcal{U}_{-\rho}, \mathcal{U}_{-\sigma}]_{x-n_\rho-n_\sigma,x} \right. \\ \left. - i[\mathcal{U}_{+\mu}, \lambda_\mu]_{x,x-a}(\rho)_{x-a,x} - i(\tilde{\rho})_{x,x+\tilde{a}} \epsilon_{\mu\nu} [\mathcal{U}_{-\mu}, \lambda_\nu]_{x+\tilde{a},x} \right]$$

$$(\mathcal{U}_{\pm\mu})_{x\pm n_\mu,x} = (e^{\pm i(A_\mu \pm i\phi^{(\mu)})})_{x\pm n_\mu,x}$$



Gauge invariance is kept ?

$$(s_A \varphi)_{x+a_\varphi+a_A,x} = s_A(\varphi)_{x+a_\varphi,x} \equiv [\nabla_A, \varphi]_{x+a_\varphi+a_A,x}$$

s_A -operation to the action makes a fermionic link hole $[x, x + a_A]$

$$\begin{array}{ccc}
 [S_{LSYM}]_{x,x} & \xrightarrow{s_A} & (s_A[S_{LSYM}])_{x+a_A,x} \\
 & \downarrow \xi^A \text{ (shift : } -a_A) & \\
 & [\xi^A(s_A[S_{LSYM}])]_{x,x} &
 \end{array}
 \begin{array}{c}
 \text{SUSY tr.} \\
 \downarrow \\
 \text{SUSY var.}
 \end{array}$$

This last step is possible only when

$$[\xi^A, \varphi]_{x-a_A+a_\varphi,x} = 0 \quad \text{for all } \varphi \left(\begin{array}{l} \xi_A = \text{constant fermionic} \\ \text{link parameter} \end{array} \right.$$

SUSY variation

$$\delta_A S(\varphi) \equiv S(\varphi + \delta_A \varphi) - S(\varphi)$$

$$\begin{aligned} (\delta_A \varphi)_{x+a_\varphi, x} &\equiv (\xi_A)_{x+a_\varphi, x+a_\varphi+a_A} (S_A \varphi)_{x+a_\varphi+a_A, x} \\ &= (\xi_A)_{x+a_\varphi, x+a_\varphi+a_A} S_A(\varphi)_{x+a_\varphi, x} \\ &= (\xi_A)_{x+a_\varphi, x+a_\varphi+a_A} [\nabla_A, \varphi]_{x+a_\varphi+a_A, x} \end{aligned}$$

link super parameter:

$$(\xi_A)_{x-a_A, x} \equiv e^{-a_A \cdot \partial} \xi_A, \quad \text{with} \quad [x_\nu, \xi_A] = 0$$

anti-commute with all the super connections:

$$(\xi_A)_{x+a_B-a_A, x+a_B} (\nabla_B)_{x+a_B, x} + (\nabla_B)_{x+a_B-a_A, x-a_A} (\xi_A)_{x-a_A, x} = 0$$

thus (anti-)commute with all the fields φ which are made of ∇_A

$$[\xi_A, \varphi] = 0$$

gauge invariant:

$$\begin{aligned} (\xi_A)_{x-a_A, x} &\rightarrow G^{-1}(x-a_A) (\xi_A)_{x-a_A, x} G(x) \\ &= (\xi_A)_{x-a_A, x} G^{-1}(x) G(x) \\ &= (\xi_A)_{x-a_A, x} \end{aligned}$$

surface terms cancellation

In the process of proving SUSY invariance of the action we sometimes need the following symbolic relations:

$$\sum_x = \sum_{x+n_\mu}, \quad \text{or} \quad \sum_x = \sum_{x+a_A}$$

The first relation is trivial since x is lattice site of square lattice.

For example

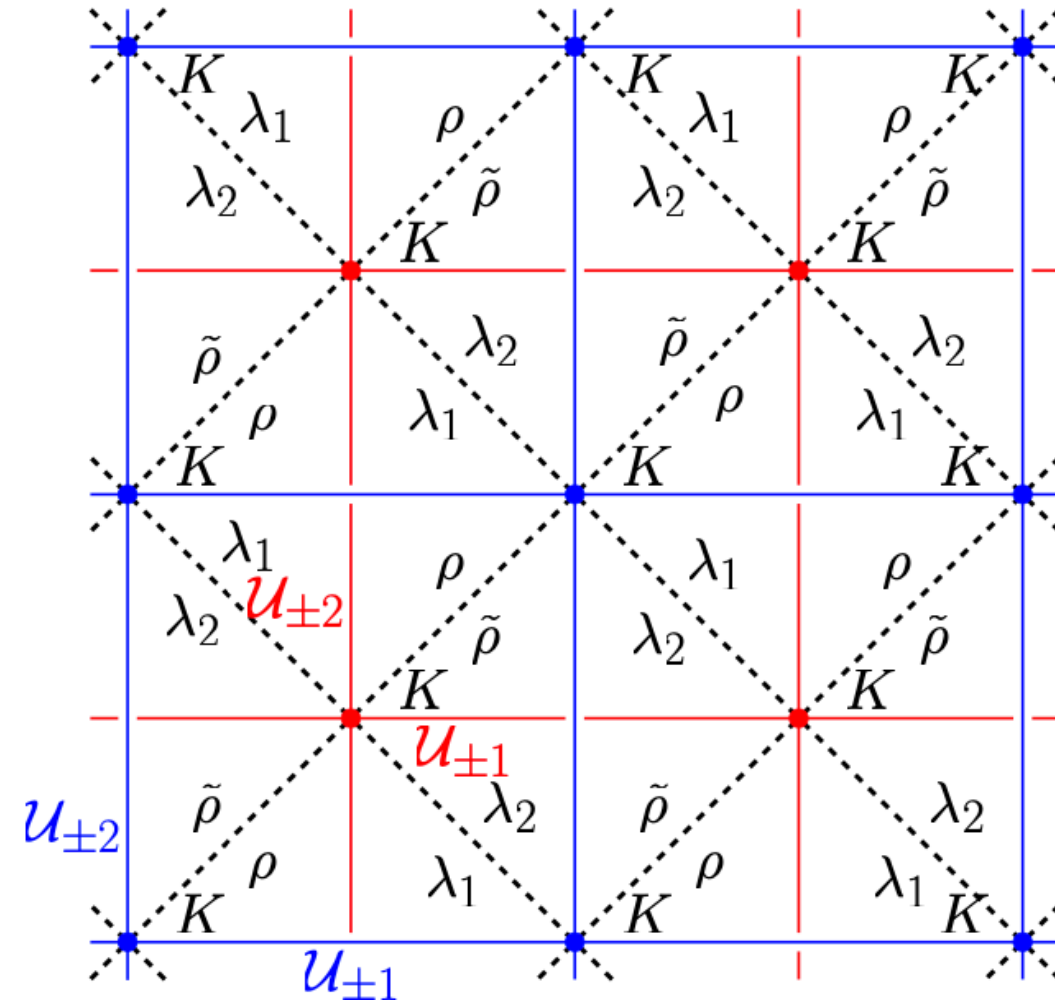
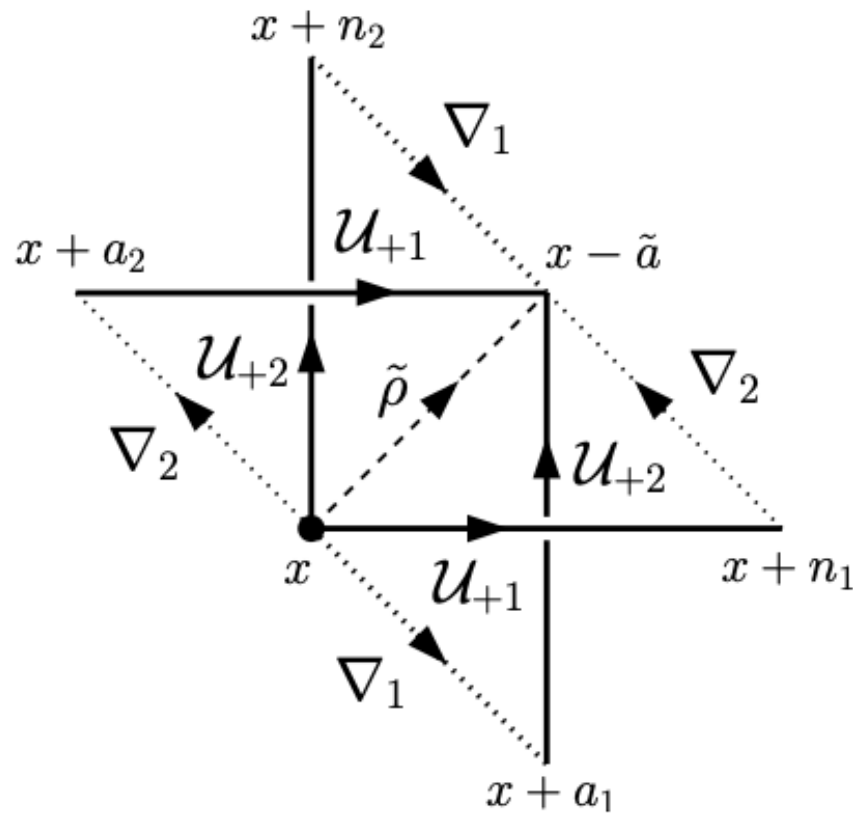
$$\sum_x \{ (\Phi_1)_{x,x+a_A} (\Phi_2)_{x+a_A,x} - (\Phi_1)_{x+a_B,x+a_B+a_A} (\Phi_2)_{x+a_B+a_A,x+a_B} \} = 0$$

$$\text{If} \quad \sum_x = \sum_{x+a_B}$$

	∇	$\tilde{\nabla}$	∇_μ	$\mathcal{U}_{\pm\mu}$
shift	a	$\tilde{a}$	a_μ	$\pm n_\mu$

	ρ	$\tilde{\rho}$	λ_μ	K
shift	$-a$	$-\tilde{a}$	$-a_\mu$	0

Double space time structure of lattice super Yang-Mills $a = (\frac{1}{2}, \frac{1}{2})$



$$[\nabla_\mu, \mathcal{U}_{+\nu}]_{x+a_\mu+n_\nu, x} \equiv -\epsilon_{\mu\nu}(\tilde{\rho})_{x-\tilde{a}, x},$$

$$[\nabla_\mu, \mathcal{U}_{-\nu}]_{x+a_\mu-n_\nu, x} \equiv -\delta_{\mu\nu}(\rho)_{x-a, x},$$

$$\epsilon_{\mu\nu}[\nabla, \mathcal{U}_{-\nu}]_{x+a-n_\nu, x} = -[\tilde{\nabla}, \mathcal{U}_{+\mu}]_{x+\tilde{a}+n_\mu, x} \equiv -\epsilon_{\mu\nu}(\lambda_\nu)_{x-a_\nu, x},$$

shift vector relations

$$\left. \begin{aligned} a + a_\mu &= n_\mu, & \tilde{a} + a_\mu &= -|\epsilon_{\mu\nu}|n_\nu \\ a + \tilde{a} + a_1 + a_2 &= 0 \end{aligned} \right\} \longleftrightarrow \left\{ \begin{aligned} a_1 &= -a + n_1, & a_2 &= -a + n_2, \\ \tilde{a} &= a - n_1 - n_2 \end{aligned} \right.$$

	∇	$\tilde{\nabla}$	∇_μ	$\mathcal{U}_{\pm\mu}$
shift	a	$\tilde{a}$	a_μ	$\pm n_\mu$

	ρ	$\tilde{\rho}$	λ_μ	K
shift	$-a$	$-\tilde{a}$	$-a_\mu$	0

define:

$$\Lambda = \{m_1 \hat{n}_1 + m_2 \hat{n}_2 | m_1, m_2 \in \mathbb{Z}\}$$

$$\Lambda_a = U_{j \in \mathbb{Z}}(\Lambda + ja)$$

define: $X = \Lambda_a - \frac{a}{2} = \Lambda_a + \frac{a}{2},$

mid point of fermion $\phi : r_\phi = x + \frac{a_\phi}{2}$

closure:

$$\Lambda_a + a = \Lambda_a \quad (\Sigma_x = \Sigma_{x+a_A})$$

$$\begin{aligned} r_\rho &\in X & r_{\lambda_1} &\in X - \frac{n_1}{2} \\ r_{\tilde{\rho}} &\in X + \frac{(n_1 + n_2)}{2} & r_{\lambda_2} &\in X - \frac{n_2}{2} \end{aligned}$$

$$r_\rho \in X \quad r_{\lambda_1} \in X - \frac{n_1}{2}$$

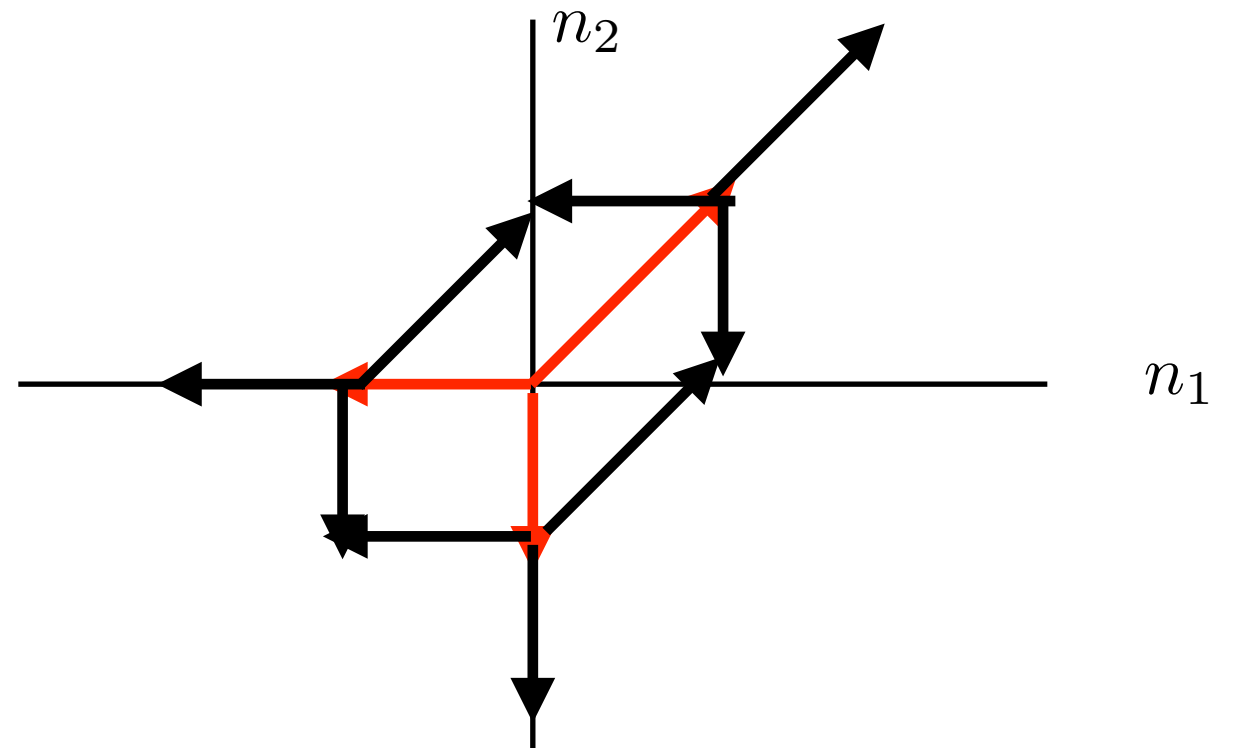
$$r_{\tilde{\rho}} \in X + \frac{(n_1 + n_2)}{2} \quad r_{\lambda_2} \in X - \frac{n_2}{2}$$

The relative distance of 4 fermions is always fixed as

$$0, \quad -\frac{n_1}{2}, \quad -\frac{n_2}{2}, \quad \frac{n_1 + n_2}{2}$$

square lattice of half lattice constant

$$x_n = \frac{na}{2} + \frac{a}{4} \quad (a: \text{lattice constant})$$

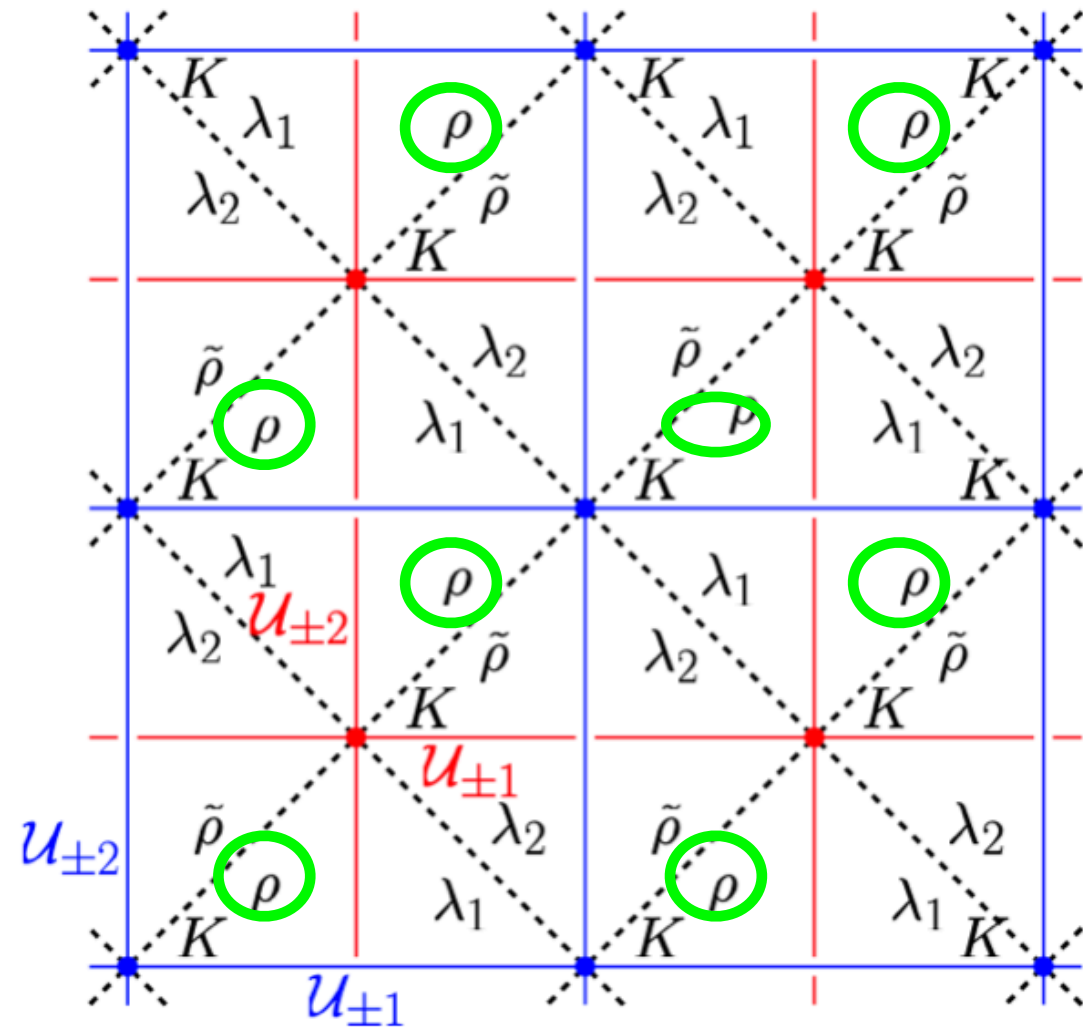


The basic half lattice cycle structure is hidden here !

n_1 coordinate of ρ

$$x = \frac{na}{2} + \frac{a}{4}$$

lattice constant a



momentum representation of fermion field for n_1 direction

$$\psi(p) = \frac{1}{2} \sum_{x=\frac{na}{2} + \frac{a}{4}} e^{ipx} \psi(x) = -\psi(p \pm \frac{4\pi}{a})$$

n_1 component of ρ

anti-periodic with respect to $\frac{4\pi}{a}$

Two zero modes; translational invariant with lattice constant

$$\Phi_1(x) = c_1, \quad \Phi_2(x) = (-1)^n c_2, \quad x = \frac{na}{2} + \frac{a}{4}$$

fermion field with alternating sign

$$\frac{1}{2} \sum_{x=\frac{na}{2}+\frac{a}{4}} e^{ipx} (-1)^{\frac{2x}{a}} \psi(x) = \psi\left(p + \frac{2\pi}{a}\right) = -\psi\left(p - \frac{2\pi}{a}\right)$$

$$(-1)^{\frac{2x}{a}} = e^{\pm \frac{2\pi i x}{a}}$$

$\psi(p + \frac{2\pi}{a})$ is half cycle shift momentum of $\psi(p)$

This fermion field is the corresponding species doubler field.

n_1 component of $\tilde{\rho}$

species doublers as twisted fermions

We treat similarly n_2 direction. Then we can assign
 $2 \times 2 = 4$ chiral fermion species doublers to

$$\rho, \tilde{\rho}, \lambda_1, \lambda_2$$

corresponding zero mode: $1, (-1)^{n_1+n_2}, (-1)^{n_1}, (-1)^{n_2}$

momentum shift: $(0, 0), (\frac{2\pi}{a}, 0), (0, \frac{2\pi}{a}), (\frac{2\pi}{a}, \frac{2\pi}{a})$

In order to realize exact SUSY of N=D=2 on the lattice
1 cycle $= a/2$ lattice is needed.

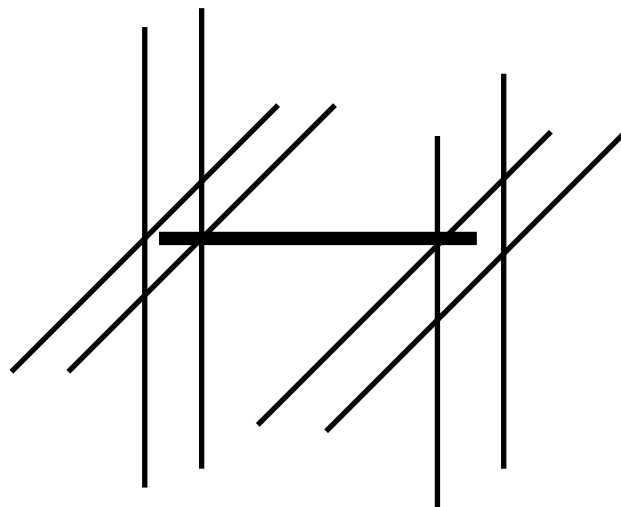
typical N=2 SUSY invariant lattice space

one dimension hidden in the arbitrariness of a

Λ_a can be considered as defining extra dimension by identifying

$$j = n_0$$

Two two-dimensional lattices which are dual to each other
connected by link fermions constitute the space:



D=N=2:

D=N=4:

scale invariance of SUSY algebra

$$\nabla_{\pm\mu} = \partial_{\mu} - i\Gamma_{\mu}^{\pm}$$

continuum twisted SUSY algebra: $\{\nabla_A, \nabla_B\} = f_{AB}^{\mu} \nabla_{\pm\mu}$

scale dimension: $[\mathbf{D}, \nabla_A] = \frac{1}{2} \nabla_A, \quad [\mathbf{D}, \nabla_B] = \frac{1}{2} \nabla_B, \quad [\mathbf{D}, \nabla_{\pm\mu}] = \nabla_{\pm\mu}$

scale invariance:

$$\begin{aligned} [D, \{\nabla_A, \nabla_B\}] &= \{[D, \nabla_A], \nabla_B\} + \{\nabla_A, [D, \nabla_B]\} = \{\nabla_A, \nabla_B\} = f_{AB}^{\mu} \nabla_{\pm\mu} \\ &\parallel \\ &f_{AB}^{\mu} \nabla_{\pm\mu} \end{aligned}$$

On the lattice what happens ?

lattice twisted SUSY algebra: $\{\nabla_A, \nabla_B\} = -f_{AB}^{\mu} e^{-\nabla_{+\mu}}$

new algebra

scale dimension ?: $[\mathbf{D}, \nabla_A] = d_A \nabla_A, \quad [\mathbf{D}, \nabla_B] = d_B \nabla_B$

$$[\mathbf{D}, \nabla_A] = d_A \nabla_A, \quad [\mathbf{D}, \nabla_B] = d_B \nabla_B \quad \{\nabla_A, \nabla_B\} = -f_{AB}^\mu e^{-\nabla_{+\mu}}$$

$$e^{\mathbf{D}} \{\nabla_A, \nabla_B\} = e^{d_A + d_B} \{\nabla_A, \nabla_B\} = -f_{AB}^\mu e^{\mathbf{D}} e^{-\nabla_{+\mu}} e^{-\mathbf{D}}$$

$$\sim e^{-\nabla_{+\mu}}$$

$$\parallel$$

$$\begin{aligned} e^{\mathbf{D}} \{\nabla_A, \nabla_B\} &= \sum_n \frac{1}{n!} [\mathbf{D} [\mathbf{D} [\cdots [\mathbf{D}, \overline{\{\nabla_A, \nabla_B\}}] \cdots]] \sim e^{\mathbf{D}} e^{-\nabla_{+\mu}} e^{-\mathbf{D}} \\ &= \sum_n \frac{1}{n!} (d_A + d_B)^n \{\nabla_A, \nabla_B\} = e^{d_A + d_B} \{\nabla_A, \nabla_B\} \end{aligned}$$

$$\underline{[\mathbf{D}, \{\nabla_A, \nabla_B\}]} = \{\nabla_A [\mathbf{D}, \nabla_B]\} + \{\nabla_B [\mathbf{D}, \nabla_A]\} = (d_A + d_B) \{\nabla_A, \nabla_B\}$$

$$e^{d_A+d_B} \{ \nabla_A, \nabla_B \} = -f_{AB}^\mu e^{\mathbf{D}} e^{-\nabla_{+\mu}} e^{-\mathbf{D}}$$

||

$$e^{(d_A+d_B)} (-f_{AB}^\mu e^{-\nabla_{+\mu}})$$

$$e^{\mathbf{D}} e^{-\nabla_{+\mu}} = e^{d_A+d_B} e^{-\nabla_{+\mu}} e^{\mathbf{D}}$$

$$\nabla_{+\mu} = \partial_\mu - i\Gamma_\mu^+$$

Weyl-'t Hooft algebra

Switch off the gauge fields we find a special solution !

$$e^{\mathbf{D}} e^{-\partial_\mu} = e^{d_A+d_B} e^{-\partial_\mu} e^{\mathbf{D}}, \quad \text{for } e^{-\nabla_{+\mu}}$$

$$[\partial_\mu, \mathbf{D}] = d_A + d_B, \quad \text{for } e^{-\nabla_{+\mu}}$$

$$(d_A + d_B)_\nu = +(n_\mu)_\nu, \quad \text{for } e^{-\nabla_{+\mu}}$$

if we identify: $d_A = a_A, \quad d_B = a_B$

$$[\partial_\mu, \mathbf{D}_\nu] = (n_\nu)_\mu = \delta_{\nu\mu} \longrightarrow \mathbf{D}_\nu = x_\nu$$

“canonical
commutation relation”

non-commutative algebra

$$[x_\nu, \nabla_A] = (a_A)_\nu \nabla_A, \quad \longleftarrow [\mathbf{D}, \nabla_A] = d_A \nabla_A$$

$$e^{x_\nu} e^{-\nabla_{+\mu}} = e^{\delta_{\nu\mu}} e^{-\nabla_{+\mu}} e^{x_\nu}$$

switch off the gauge fields: $\nabla_A \rightarrow Q_A = \frac{\partial}{\partial \theta_A} - \frac{1}{2} f_{AB}^\mu \theta_B e^{-\partial_\mu}$

$$\left[x_\nu, \frac{\partial}{\partial \theta_A} - \frac{1}{2} f_{AB}^\mu \theta_B e^{-\partial_\mu} \right] = (a_A)_\nu \left(\frac{\partial}{\partial \theta_A} - \frac{1}{2} f_{AB}^\mu \theta_B e^{-\partial_\mu} \right) \quad \text{for } a_A + a_B = +n_\mu$$

$$\left[x_\nu, \frac{\partial}{\partial \theta_A} \right] = (a_A)_\nu \frac{\partial}{\partial \theta_A}, \quad [x_\nu, \theta_A] = -(a_A)_\nu \theta_A \quad \leftrightarrow \quad (x + a_A)_\nu \theta_A = \theta_A x_\nu$$

non-commutative

$$(\nabla_A)_{x+a_A, x} = e^{+a_A \cdot \partial} \nabla_A(x) \quad [x_\nu, e^{+a_A \cdot \partial} \nabla_A(x)] = 0$$

$$([x_\nu, e^{a_A \cdot \partial}] = -(a_A)_\nu e^{a_A \cdot \partial})$$

$$\begin{aligned}
[x_\nu, \nabla_A] &= (a_A)_\nu \nabla_A, \longrightarrow [x_\nu, Q_A] = (a_A)_\nu Q_A \\
&\longrightarrow (x - a_A)_\nu Q_A = Q_A x_\nu \\
&\longrightarrow \{(x - a_A)_\nu\}^k Q_A = Q_A \{x_\nu\}^k \\
&\longrightarrow \phi_i(x - a_A) Q_A = Q_A \phi_i(x)
\end{aligned}$$

$$a_A \longrightarrow -a_A$$

$$\begin{aligned}
Q_A(\phi_1(x)\phi_2(x)) &= (Q_A\phi_1(x))\phi_2(x) + \phi_1(x + a_A)Q_A\phi_2(x) \\
&\parallel \\
Q_A(\phi_2(x)\phi_1(x)) &= (Q_A\phi_2(x))\phi_1(x) + \phi_2(x + a_A)Q_A\phi_1(x)
\end{aligned}$$

The origin of the shift of lattice Leibniz rule of super charge.

Summary

- N=D=2, N=D=4 twisted super Yang-Mills are formulated on a lattice exactly keeping super symmetry for all super charges.
- Algebraic structure of graded Jacobi identities and shift vector relations determine the structure of two dimensional lattice space: Doubled two-dimensional square lattice spaces are generated and they are connected by link fermions.
- Half lattice distance periodicity of twisted fermions is established and they can be identified as species doublers of chiral fermions.

Algebraic consistency determines the space time structure.

- In the exponentialized new lattice algebra we can derive Weyl-'t Hooft relation. Non-gauge version of this relation leads non commutativity of fermionic and bosonic coordinate: $\theta_A x_\nu = (x + a_A)_\nu \theta_A$