石橋-川合の弦の場の理論と位相的漸化式について

Hiroyuki FUJI

Center for Mathematical and Data Sciences & Department of Mathematics,
Kobe University

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Based on works in collaborations with Masahide Manabe and Yoshiyuki Watabiki:
"Dynamical Triangulations for 2D Pure Gravity and Topological Recursion," JHEP **05** (2026) 208.
"Multicritical Dynamical Triangulations and Topological Recursion," arXiv:2512.10519 [hep-th].
"A Hamiltonian Formalism for Topological Recursion," JHEP **08** (2026) 081.

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1. イントロダクション

CEO 位相的漸化式とは？

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- エルミート型行列模型に対するレゾルベントの連結相関関数に対する Schwinger-Dyson 方程式の $1/N$ 展開から得られる漸化式として発見

$$\left\langle \mathrm{Tr}\frac{1}{x_1 - M}\cdots\mathrm{Tr}\frac{1}{x_n - M}\right\rangle_N^{\mathrm{conn}} = \sum_{g\geq 0} N^{2-2g-n} W_{g,n}(x_1,\ldots,x_n)$$

- 漸化式はスペクトル曲線のデータで決定される：

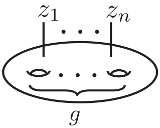
  $X = \{(x, y) \in \mathbb{C}^2 | y^2 = \prod_i (x - c_i)\}$

  $\Rightarrow$ 行列模型が無くてもスペクトル曲線のデータがあれば定められる．

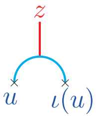
  相関関数を効率良く計算するためのツールというよりも，相関関数の背後に潜む代数・幾何的構造を浮き彫りにするための枠組み．

CEO 位相的漸化式のグラフィカル表示

- 漸化式の構成要素 $K(z,u) := \dfrac{\int_{z'=\iota(u)}^{z'=u} B(z,z')}{2(y(u)-y(\iota(u)))dx(u)/du}$

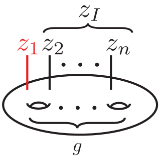


$\leftrightarrow \omega_{g,n}(z_1,\dots,z_n)$



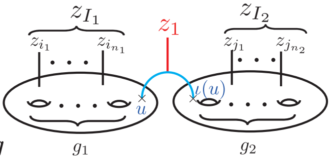
$$\leftrightarrow \sum_{\substack{w:\text{branch pt.}\\ x(\iota(w))=x(w)}} \operatorname*{Res}_{u\to w} K(z,u)$$

- CEO 位相的漸化式のグラフィカル表示：



$$= \sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=I}}$$

$I=\{2,\dots,n\}$



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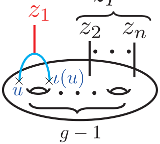


位相的漸化式に関する 3 つ視点

位相的漸化式にはいくつかの見方がある：

- 行列模型 (Eynard のオリジナルアイデア):

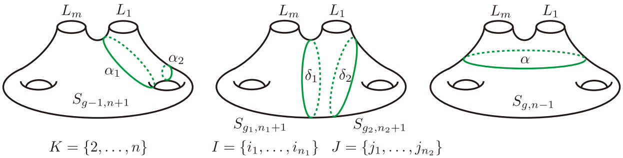
  ⇒ Schwinger-Dyson 方程式 (Abstract loop eq.)

- 双曲幾何学 (Laplace 双対):

  ⇒ Mizakhani の漸化式 (幾何学的漸化式)

- 代数・表現論:

  ⇒ Virasoro 拘束条件 (量子エアリ構造)



石橋-川合の非臨界弦の場の理論

2 次元ミニマル重力の非摂動的定式化 ⇒ 2 つの鍵となる研究

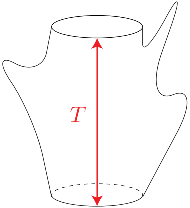
- Schwinger-Dyson 方程式と Virasoro 拘束条件

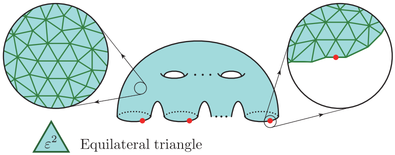
  [福間-川合-中山 91, Dijkgraaf-Verlinde-Verlinde91]

- Dynamical triangulation (DT) による 2 次元重力の転送行列の解析

  [川合-河本-最上-綿引 93]

⇒ 石橋-川合の非臨界弦の場の理論（弦の相互作用が入った Hamiltonian）



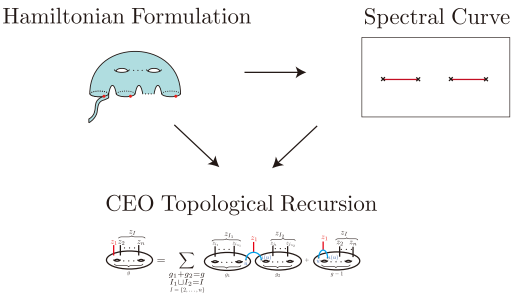


本日の内容

本研究の１つの goal: 逆問題

CEO 位相的漸化式から対応する石橋-川合の弦の場の理論を見出す．

- スペクトル曲線 $\mathcal{C} = (X, (x, y), B) \rightarrow$ 弦の場の理論の Hamiltonian
- 2 次元重力理論に限らず，CEO 位相的漸化式に従うシステムに適用．



2. DT Hamiltonian と CEO 位相的漸化式

Dynamical triangulation

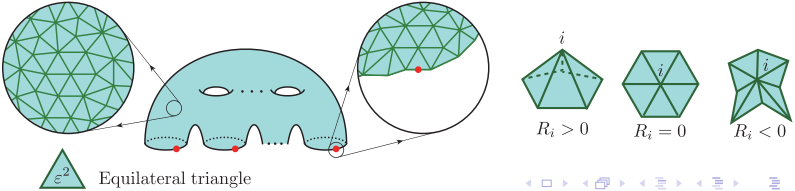
Discretized model of 2-dim. pure quantum gravity

$\Rightarrow$ Enumerative problem for # of triangulation of connected 2D surfaces

with the fixed boundary lengths $\ell_1, \ldots, \ell_n \in \mathbb{N}$ (N_2: # triangles)

$$\longrightarrow F_n^{(g)}(\ell_1, \ldots, \ell_n) = \sum_{N_2 \geq 1} \sum_{\mathcal{T}_n^{(g)}(\ell_1,\ldots,\ell_n;N_2)} \kappa^{N_2}.$$

$\mathcal{T}_n^{(g)}(\ell_1, \ldots, \ell_n; N_2)$: Triangulations w/ geometric data $(g; \ell_1, \ldots, \ell_n; N_2)$.



Operator formalism

Peeling decomposition $\to$ Deformation of a boundary

$\Rightarrow$ A nice representation by boundary creation / annihilation operators:

- Creation / annihilation operators of a boundary with length ℓ:

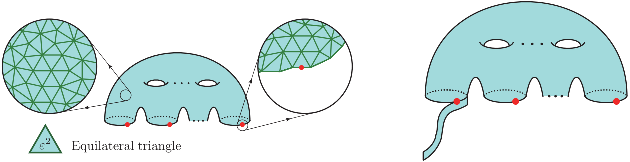
$$[\Psi(\ell), \Psi^\dagger(\ell')] = \delta_{\ell,\ell'}, \quad [\Psi(\ell), \Psi(\ell')] = 0 = [\Psi^\dagger(\ell), \Psi^\dagger(\ell')]$$

- Vacuum state $|\mathrm{vac}\rangle$: $\Psi(\ell)|\mathrm{vac}\rangle = 0$

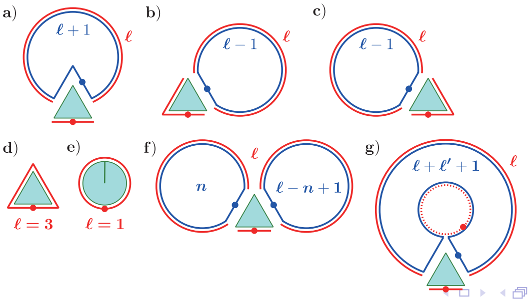
- Multi-loop state: $\Psi^\dagger(\ell_1)\Psi^\dagger(\ell_2)\cdots\Psi^\dagger(\ell_n)|\mathrm{vac}\rangle$

Peeling decomposition

Peeling decomposition: eliminate a marked triangle:



7 types of decompositions:



Hamiltonian operator

Definition 2.1

We define a Hamiltonian operator $\mathcal{H}$:

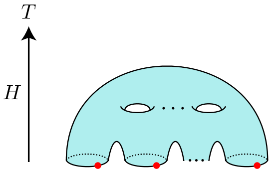
$$\begin{aligned}\mathcal{H} = &\sum_{\ell=1}^{\infty} \Psi^\dagger(\ell)\ell\Psi(\ell) - \kappa\sum_{\ell=1}^{\infty} \Psi^\dagger(\ell+1)\ell\Psi(\ell) - 2\kappa\sum_{\ell=2}^{\infty} \Psi^\dagger(\ell-1)\ell\Psi(\ell)\\ &- 3\kappa\Psi(3) - \kappa\Psi(1)\\ &- \kappa\sum_{\ell=1}^{\infty}\sum_{n=1}^{\ell} \Psi^\dagger(n)\Psi^\dagger(\ell-n+1)\ell\Psi(\ell)\\ &- G\kappa\sum_{\ell=1}^{\infty}\sum_{\ell'=1}^{\infty} \Psi^\dagger(\ell+\ell'+1)\ell'\Psi(\ell')\ell\Psi(\ell).\end{aligned}$$

Acting $\mathcal{H}$ on a multi-loop state $\Rightarrow$ one step decomposition by the peeling.

石橋-川合のハミルトニアンと Schwinger-Dyson 方程式

- ミニマル重力の分配関数 $F_n(\ell_1,\ldots,\ell_n;G)$，ループ生成演算子 $\Psi^\dagger(\ell)$

$$F_n(\ell_1,\ldots,\ell_n;G) = \lim_{T\to\infty}\langle\mathrm{vac}|\mathrm{e}^{-T\mathcal{H}}\Psi^\dagger(\ell_1)\cdots\Psi^\dagger(\ell_n)|\mathrm{vac}\rangle.$$



- Schwinger-Dyson 方程式

$$\begin{aligned}0 &= \lim_{T\to\infty}\frac{\partial}{\partial T}\langle\mathrm{vac}|\mathrm{e}^{-T\mathcal{H}}\Psi^\dagger(\ell_1)\cdots\Psi^\dagger(\ell_n)|\mathrm{vac}\rangle \\ &= \lim_{T\to\infty}\langle\mathrm{vac}|\mathrm{e}^{-T\mathcal{H}}\mathcal{H}\Psi^\dagger(\ell_1)\cdots\Psi^\dagger(\ell_n)|\mathrm{vac}\rangle.\end{aligned}$$

CEO topological recursion for the DT model

Theorem 2.2 (F-Manabe-Watabiki)

$\widetilde{F}_n^{(g)}(x_1,\ldots,x_n)$ *obeys the CEO topological recursion with the spectral curve* $\mathcal{C}=(\mathbb{CP}^1,x,y,B)$ *such that*

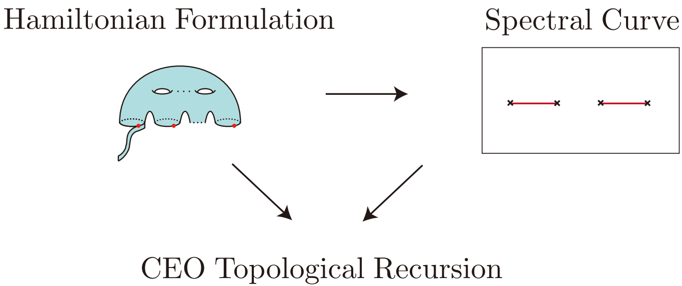
$$y(x)=\frac{1}{2x^3}\left(x-\frac{c}{\kappa}\right)\sqrt{x\left(x-\frac{4\kappa}{c^2}\right)},\quad c(1-c^2)=8\kappa^2,$$

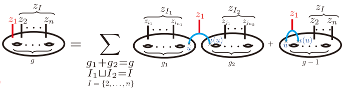
$$x(z)=\frac{\kappa}{c^2}\left(2+z+\frac{1}{z}\right),\quad B(z_1,z_2)=\frac{dz_1\otimes dz_2}{(z_1-z_2)^2}.$$

This CEO topological recursion is found from the Schwinger-Dyson equation found from the Hamiltonian formalism.

3. スペクトル曲線データからハミルトニアンの逆構成

Summary & Graphical abstract





Inverse problem:

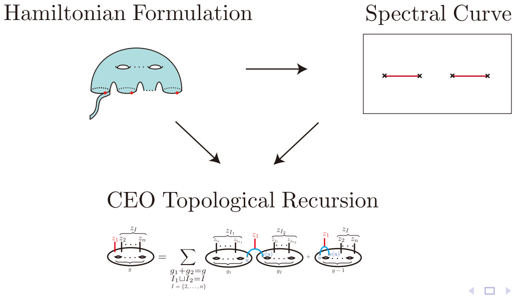
Can we reconstruct the Hamiltonian $\mathcal{H}$ from the spectral curve data?

Spectral curve data

Our ansatz for the spectral curve:

$$y = M(x)\sqrt{\sigma(x)}, \quad M(x) = q(x)/p(x), \quad \sigma(x) = \prod_{k=1}^{br(h)} (x - \alpha_k)$$

$$p(x) = \sum_{k=\mu}^{r} p_k x^k, \ (0 \le \mu \le r), \quad q(x) = \sum_{k=0}^{s-h} q_k x^k, \ (h \le s).$$



Ambjørn-Watabiki's recipe for finding Hamiltonian

(1) Prepare the $W_n^{(3)}$ generator in free field representation:

- $br(h) = 2h + 2$: even $\Rightarrow W_n^{(3)} = \frac{1}{3}\sum_{k+l+m=n} :a_k a_l a_m:$
- $br(h) = 2h + 1$: odd $\Rightarrow W_n^{(3)} = \frac{1}{4}\left(\frac{1}{3}\sum_{k+l+m=2n} :a_k a_l a_m: + \frac{1}{4}a_{2n}\right)$

$[a_m, a_n] = m a_{m+n,0}, \quad [a_n, W_{-2}^{(3)}] = 2nL_{n-2} \quad (n = 1, 2, 3, \cdots)$

$\Rightarrow$ Witt alg. $[L_m, L_n] = (m-n)L_n \ (m, n \geq -1)$

(2) Consider Y of the linear terms of a_n $(n \geq 0)$

$\Rightarrow \mathcal{H} = -2\sqrt{g_s}W_{-2}^{(3)} + Y$ such that $\mathcal{H}|\mathrm{vac}\rangle = 0$.

(3) Identify the loop operators $\psi(n), \psi^\dagger(n)$ and a's:

$a_n \to \sqrt{c/g_s}\Psi^\dagger(n)$, $a_{-n} \to n(\sqrt{g_s/c}\Psi(n) + \lambda_n)$ $(n > 0)$, $a_0 = \lambda_0$

Examples

$br(h) = 2h + 2$ case:

- 1-Matrix model (e.g. Penner model)
- Discrete DT models
- 2D CDT models

$br(h) = 2h + 1$ case:

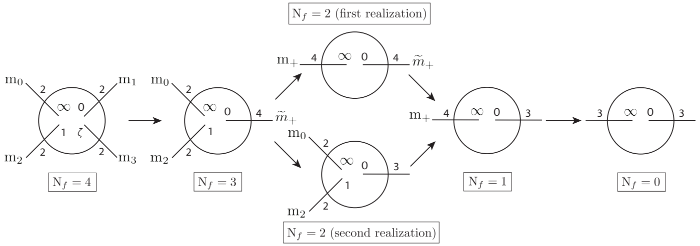
- $(2, 2m-1)$ minimal string
- Topological gravity, JT gravity
- Seiberg-Witten theory for 4D $\mathcal{N} = 2$ SYM

4. Seiberg-Witten 曲線の場合

Example: Degeneration of Seiberg-Witten curves

Degeneration of $SU(2)$ Seiberg-Witten curve (N_f flavor hypermultiplets) [Gaiotto'09][Eguchi-Maruyoshi'09][Kozcaz-Pasquetti-Wyllard'10]

Topological recursion for SW curve [Awata-Fuji-Kanno-Manabe-Yamada'10]



Example: Seiberg-Witten curve for $N_f = 4$ flavors :

$$y^2 = \left(\frac{m_0}{x(x-1)(x-\zeta)}\right)^2 (x^4 + S_1 x^3 + S_2 x^2 + S_3 x + S_4).$$

Hamiltonian for SW curve for $N_f = 1$

- Seiberg-Witten curve for $N_f = 1$ flavors :

$$y^2 = \frac{\Lambda_1^2}{x^3} + \frac{u}{x^2} + \frac{m_+\Lambda_1}{x} + \frac{\Lambda_1^2}{4} = \frac{\Lambda_1^2}{4x^4}\left(x^4 + \frac{4m_+}{\Lambda_1}x^3 + \frac{4u}{\Lambda_1^2}x^2 + 4x\right).$$

- Hamiltonian:

$$-\mathcal{H}_{N_f=1}^{SU(2)} = g_s^{-1}\Lambda_1^2\,\Psi(1) + 2\sum_{k=2,3}\tau_k\sum_{\ell\geq 1}\ell\,\Psi^\dagger(\ell+k-2)\Psi(\ell)$$

$$-\,g_s\sum_{\ell,\ell'\geq 1}\left((\ell+\ell')\,\Psi^\dagger(\ell)\Psi^\dagger(\ell')\Psi(\ell+\ell') + \ell\ell'\,\Psi^\dagger(\ell+\ell')\Psi(\ell)\Psi(\ell')\right).$$

Parameters:

$$\tau_2 = -m_+\,,\quad \tau_3 = -\frac{1}{2}\Lambda_1\,,$$

$$\Lambda_1\lim_{g_s\to 0} g_s \lim_{T\to\infty}\left\langle \mathrm{vac}\middle|\mathrm{e}^{-T\mathcal{H}_{N_f=1}^{SU(2)}}\,\Psi^\dagger(1)\middle|\mathrm{vac}\right\rangle = u - m_+^2\,.$$

Some directions

- Spectral curve covered by more than 3 sheets [Bouchard-Eynard]

  Ising model : 3-sheeted spectral curve

  $\Rightarrow$ Hamiltonian from $W^{(4)}$ generator [Ambjørn-Watabiki]

  cf. [Ikehara-Ishibashi-Kawai-Mogami-Nakayama-Sasakura]

- Thin spectral curve [Zhou]

  $O(N)$ vector model [Nishigaki-Yoneya][Di Vecchia-Kato-Ohta]

  $\Rightarrow$ Spectral curve with no branched point?

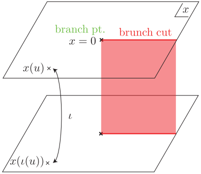
- Normal matrix model [Zabrodin-Wiegmann]

  Droplet distribution of eigenvalues (cf. [Lin-Lunin-Maldacena])

Appendix

CEO 位相的漸化式の復習

Basic data of the CEO topological recursion



Spectral curve: $\mathcal{C} = (C, \mathrm{x}, \mathrm{y}, B)$

- C: algebraic curve $C = \{x, y \in \mathbb{C} | y^2 = f(x)\}$

  Simplest example: Airy curve $y^2 = -x$

- (x, y): Coordinate functions

  Airy case: $\mathrm{x}(z) = -z^2, \; \mathrm{y}(z) = -z$

  $z \in \mathbb{CP}^1$: local coordinate near branch pts. $x = x(\zeta_p)$

  $(p = 1, \ldots, \#\text{branch pts.})$

- B: A meromorphic bi-differential on $C^{\times 2}$

  Airy case: $B(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} dz_1 \otimes dz_2$

CEO 位相的漸化式

CEO 位相的漸化式 [Chekhov-Eynard-Orantin06, Eynard-Orantin07]

スペクトル曲線：$\mathcal{C} = (X, (\mathrm{x}(z), \mathrm{y}(z)), B(z, z')) \quad (z_I := (z_2, \ldots, z_n))$

$$\omega_{g,n}(z_1, \ldots, z_n)$$

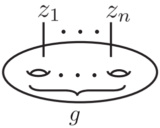
$$= \sum_{p=1}^{\#Branch\ pts.} \operatorname*{Res}_{u \to \zeta_p} K(z_1, u) \Bigg[\sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = I = \{2,\ldots,n\}}} \omega_{g_1,|I_1|+1}(u, \mathbf{z}_{I_1}) \omega_{g_2,|I_2|+1}(\iota(u), \mathbf{z}_{I_2})$$

$$+ \omega_{g-1,n+1}(u, \iota(u), \mathbf{z}_I \setminus \{z_1\}) \Bigg]$$

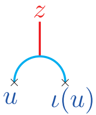
- $\omega_{g,n}(z_1, \ldots, z_n) := W_{g,n}(\mathbf{z}) dz_1 \otimes \cdots \otimes dz_n, \quad \omega_{0,2}(z_1, z_2) = B(z_1, z_2)$
- Galois involution ι: $x(\iota(u)) = x(u)$, $y(\iota(u)) = -y(u)$
- Recursion kernel: $K(z, u) = \dfrac{\int_{z'=\iota(u)}^{z'=u} B(z, z')}{2\{y(u) - y(\iota(u))\} dx(u)/du}$

CEO 位相的漸化式のグラフィカル表示

- 漸化式の構成要素 $K(z,u) := \dfrac{\int_{z'=\iota(u)}^{z'=u} B(z,z')}{2(y(u)-y(\iota(u)))dx(u)/du}$

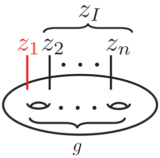


$\leftrightarrow \omega_{g,n}(z_1,\ldots,z_n)$

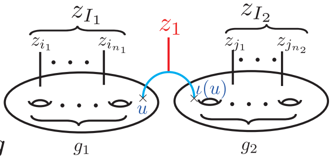


$\leftrightarrow \sum_{\substack{w:\text{branch pt.}\\ x(\iota(w))=x(w)}} \operatorname*{Res}_{u\to w} K(z,u)$

- CEO 位相的漸化式のグラフィカル表示：



$$= \sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=I\\ I=\{2,\ldots,n\}}}$$



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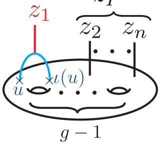
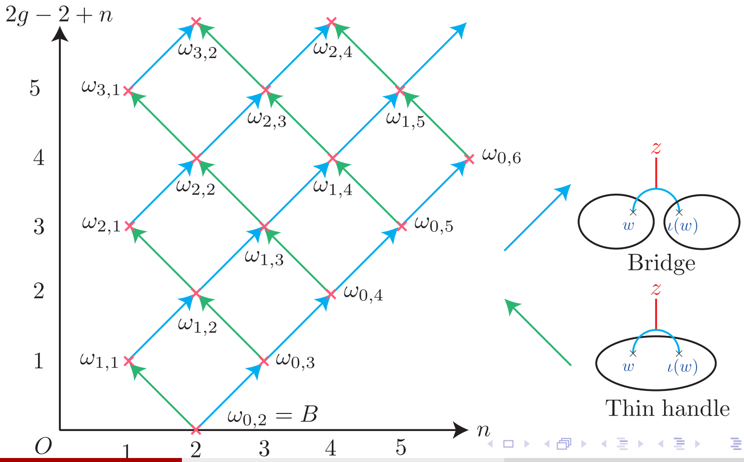


Recursive structure

Initial data: a spectral curve $\mathcal{C} := (X, (x(z), y(z)), B)$

The $W_{g,n}$s are determined recursively.



DT hamiltonian for $(2, 2m-1)$ minimal string

Hamiltonian for $(2, 2m-1)$ minimal strings

$$
\begin{aligned}
\mathcal{H}_c^{(2,2m-1)} = & -\frac{G}{4}\phi_4 + \frac{G^2}{4}(\phi_1 + \langle\phi_1\rangle)(\phi_1 + \langle\phi_1\rangle)\phi_2 \\
& - \frac{1}{2}\sum_{\ell=6}^{\infty}\sum_{n=1}^{\ell-5}\phi_n^\dagger\phi_{\ell-n-4}^\dagger \ell(\phi_\ell + \langle\phi_\ell\rangle) \\
& - \frac{G}{4}\sum_{\ell=1}^{\infty}\sum_{n=\max(5-\ell,1)}\phi_{n+\ell-4}^\dagger n(\phi_n + \langle\phi_n\rangle)(\phi_n + \langle\phi_n\rangle).
\end{aligned}
$$

where

$$
\langle\phi_n\rangle = \begin{cases} (-1)^{m-k}\frac{\tau_{2m-2k}}{(2k-1)G} & n = 2k-1 \ (1 \le k \le m+1),\ n \ne 2m-1 \\ 0 & n > m+1 \end{cases}
$$

Specialization to the Liouville gravity parameters:

$$
\tau_{2k} = \frac{(-1)^{k-1}(2m-2k-3)!!}{k!(2m-1)^{k-1}(2m-4k-1)!!}\left(\frac{(2m-1)\tau}{8}\right)^k
$$

CEO topological recursions for $(2, 2m-1)$ minimal strings

Theorem 4.1

The genus expansion of the Schwinger–Dyson equations for the $(2, 2m-1)$ DT model with $n \geq 3$ can be rewritten as the topological recursion for the spectral curve $\mathcal{C} = (X_c, (\xi, y(\xi)), B)$ determined by the disk amplitude. In particular, when the parameters are specialized to the conformal background of Liouville gravity, it can be expressed as follows:

$$y = \frac{(-1)^{m-1}\tau^{\frac{m}{2}}}{2^{m-\frac{3}{2}}} \sin\left[\frac{2m-1}{2}\arccos\left(-\frac{\xi}{\sqrt{\tau}}\right)\right], \qquad B(\xi_1, \xi_2) = \frac{d\xi_1 d\xi_2}{(\xi_1 - \xi_2)^2},$$

The same result is obtained independently in [Gregori-Schiappa21].

Hamiltonian の逆構成

Hamiltonian for $br(h)$=even

A Hamiltonian is found from the unreduced $W^{(3)}_{-2}$ generator:

$$
\begin{aligned}
-\mathcal{H}^{\mathrm{even}} = g_s^{-1}\sum_{k=1}^{\mu+2} k\,\kappa_k\,\Psi(k) + \tau_0\Psi(1)^2 + 2\sum_{k=0}^{s+2}\tau_k\sum_{\ell\geq 1}\ell\,\Psi^\dagger(\ell+k-2)\Psi(\ell) \\
+ g_s\sum_{k=\mu}^{r} p_k \sum_{\ell,\ell'\geq 1}\left(\ell+\ell'-k+2\right)\Psi^\dagger(\ell)\Psi^\dagger(\ell')\Psi(\ell+\ell'-k+2) \\
+ g_s\sum_{k=\mu}^{r} p_k \sum_{\ell,\ell'\geq 1}\ell\ell'\,\Psi^\dagger(\ell+\ell'+k-2)\Psi(\ell)\Psi(\ell')\,.
\end{aligned}
$$

Constraint eqs. $\Rightarrow (p_k, q_k, \alpha_k) \to (p_k, \kappa_k, \tau_k)$

Hamiltonian for $br(h)$=odd

A Hamiltonian is found from the 2-reduced $\overline{W}^{(3)}_{-2}$ generator:

$$
\begin{aligned}
-\mathcal{H}^{\mathrm{odd}} = {}& 2g_s^{-1}\sum_{k=1}^{\mu+1} k\,\kappa_k\,\Psi(2k) + \frac{g_s}{8}\left(2p_0\Psi(4)+p_1\Psi(2)\right) + \tau_0\,\Psi(1)\Psi(2) \\
&+ \frac{g_s p_0}{8}\Psi(1)^2\Psi(2) + 2\sum_{k=0}^{s+1}\tau_k\sum_{\ell\geq 1}\ell\,\Psi^\dagger(\ell+2k-3)\Psi(\ell) \\
&+ g_s\sum_{k=\mu}^{r}p_k\sum_{\ell,\ell'\geq 1}\left(\ell+\ell'-2k+4\right)\Psi^\dagger(\ell)\Psi^\dagger(\ell')\Psi(\ell+\ell'-2k+4) \\
&+ \frac{g_s}{4}\sum_{k=\mu}^{r}p_k\sum_{\ell,\ell'\geq 1}\ell\ell'\,\Psi^\dagger(\ell+\ell'+2k-4)\Psi(\ell)\Psi(\ell')\,.
\end{aligned}
$$

Constraint eqs. $\Rightarrow (p_k, q_k, \alpha_k) \to (p_k, \kappa_k, \tau_k)$